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Homework 4 solutions

1. A cell of some bacteria divides into two cells every 40 minutes. The initial population is 200 bacteria. Express the population after t minutes as a function of t.

Solution. Let $N(t)$ denote the amount of bacteria that is growing. Because the rate of change of the amount of substance is proportional to the amount of substance present and the initial population is 200 bacteria, we have the problem

$$\frac{dN}{dt} = kN(t),$$
$$N(0) = 200.$$

We can rewrite the separable ordinary differenital equation $\frac{dN}{dt} = kN(t)$ as

$$\frac{1}{N}\,dN = k\,dt$$

and can integrate both sides, writing

$$\int \frac{1}{N}\,dN = \int k\,dt,$$

in order to obtain

$$\ln(N) = kt + \ln(N(0)),$$

or equivalently, the explicit solution

$$N(t) = N(0)e^{kt}.$$

As the cell of some bacteria divides into two cells every 40 minutes, we have $t = 40$ minutes and $N(t) = 2N(0)$. So our solution becomes

$$2\cancel{N(0)} = \cancel{N(0)}e^{40k},$$

from which we find $k = \frac{\ln 2}{40}$. Therefore, the population of bacteria after t minutes is

$$N(t) = N(0)e^{kt} = \boxed{200e^{\frac{\ln 2}{40}t}},$$

as desired. □

2. Find the orthogonal trajectories of the family of curves $y^2 = 6cx$.

Solution. Note that $y^2 = 6cx$ is equivalent to $c = \frac{y^2}{6x}$. If we use the method of implicit differentiation, writing

$$\frac{d}{dx}(y^2) = \frac{d}{dx}(6cx)$$

we would get the differential equation

$$2y\frac{dy}{dx} = 6c = 6\frac{y^2}{6x} = \frac{y^2}{x},$$

or equivalently

$$\frac{dy}{dx} = \frac{y}{2x}$$

To find the orthogonal trajectories, we need to take the negative of the reciprocal of our original $\frac{dy}{dx}$, writing

$$\frac{dy}{dx} = -\frac{2x}{y}.$$

We can rewrite this separable equation as

$$y\,dy = -2x\,dx$$

and can integrate both sides, writing

$$\int y\,dy = \int -2x\,dx$$

in order to obtain

$$\frac{y^2}{2} = -x^2 + D,$$

or equivalently

$$\boxed{x^2 + \frac{y^2}{2} = D}$$

where D is an arbitrary constant. (I am using D to distinguish from the original c because the upper-case C looks too similar to the lower-case c.) □