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Homework 5 solutions

1. Find the general solution of
$$y'' - 6y' + 9y = 0.$$

Solution. Let $y = e^{\lambda x}$, where λ is a number. Then we obtain the first and second derivatives
$$\begin{aligned} y' &= \lambda e^{\lambda x}, \\ y'' &= \lambda^2 e^{\lambda x}. \end{aligned}$$

So we have
$$\begin{aligned} 0 &= y'' - 6y' + 9y \\ &= \lambda^2 e^{\lambda x} - 6\lambda e^{\lambda x} + 9e^{\lambda x} \\ &= e^{\lambda x}(\lambda^2 - 6\lambda + 9) \\ &= e^{\lambda x}(\lambda - 3)^2. \end{aligned}$$

Since we know $e^{\lambda x} \neq 0$, we must conclude $(\lambda - 3)^2 = 0$, which gives the repeated root $\lambda_1 = 3$. So the general solution is
$$\begin{aligned} y &= C_1 e^{\lambda_1 x} + C_2 x e^{\lambda_1 x} \\ &= \boxed{C_1 e^{3x} + C_2 x e^{3x}}, \end{aligned}$$

where C_1, C_2 are constants. □

2. Find the general solution of
$$y'' + 2y' + 2y = 0.$$

Solution. Let $y = e^{\lambda x}$, where λ is a number. Then we obtain the first and second derivatives
$$\begin{aligned} y' &= \lambda e^{\lambda x}, \\ y'' &= \lambda^2 e^{\lambda x}. \end{aligned}$$

So we have
$$\begin{aligned} 0 &= y'' + 2y' + 2y \\ &= \lambda^2 e^{\lambda x} + 2\lambda e^{\lambda x} + 2e^{\lambda x} \\ &= e^{\lambda x}(\lambda^2 + 2\lambda + 2). \end{aligned}$$

Since we know $e^{\lambda x} \neq 0$, we must conclude $\lambda^2 + 2\lambda + 2 = 0$, which gives the imaginary roots $\lambda_1 = -1 - i$ and $\lambda_2 = -1 + i$. So the general solution is
$$\begin{aligned} y &= C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x} \\ &= C_1 e^{(-1-i)x} + C_2 e^{(-1+i)x} \\ &= e^{-x}(C_1 e^{-ix} + C_2 e^{ix}) \\ &= \boxed{e^{-x}(C_1 \cos(x) + C_2 \sin(x))}, \end{aligned}$$

where C_1, C_2 are constants. □

3. Find the general solution of
$$y'''' + y'' - 6y = 0.$$

Solution. Let $y = e^{\lambda x}$, where λ is a number. Then we obtain the first, second, third, and fourth derivatives
$$\begin{aligned} y' &= \lambda e^{\lambda x}, \\ y'' &= \lambda^2 e^{\lambda x}, \\ y''' &= \lambda^3 e^{\lambda x}, \\ y'''' &= \lambda^4 e^{\lambda x}. \end{aligned}$$

So we have

$$\begin{aligned}0 &= y'''' + y'' - 6y \\ &= \lambda^4 e^{\lambda x} + \lambda^2 e^{\lambda x} - 6e^{\lambda x} \\ &= e^{\lambda x}(\lambda^4 + \lambda^2 - 6) \\ &= e^{\lambda x}(\lambda^2 - 2)(\lambda^2 + 3).\end{aligned}$$

Since we know $e^{\lambda x} \neq 0$, we must conclude $(\lambda^2 - 2)(\lambda^2 + 3) = 0$. which gives the real roots $\lambda_1 = -\sqrt{2}$ and $\lambda_2 = \sqrt{2}$ and the imaginary roots $\lambda_3 = -\sqrt{3}i$ and $\lambda_4 = \sqrt{3}i$. So the general solution is

$$\begin{aligned}y &= C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x} + C_3 e^{\lambda_3 x} + C_4 e^{\lambda_4 x} \\ &= C_1 e^{-\sqrt{2}x} + C_2 e^{\sqrt{2}x} + C_3 e^{-\sqrt{3}ix} + C_4 e^{\sqrt{3}ix} \\ &= \boxed{C_1 e^{-\sqrt{2}x} + C_2 e^{\sqrt{2}x} + C_3 \cos(\sqrt{3}x) + C_4 \sin(\sqrt{3}x)},\end{aligned}$$

where C_1, C_2, C_3, C_4 are constants. □

4. Find the Wronskian matrix of the given sets of functions and use that information to determine whether the given sets are linearly independent.

   (1) $\{x^3, x^2 + 2x\}$

   Solution. The Wronskian matrix is

$$\begin{bmatrix} x^3 & x^2 + 2x \\ \frac{d}{dx}(x^3) & \frac{d}{dx}(x^2 + 2x) \end{bmatrix} = \begin{bmatrix} x^3 & x^2 + 2x \\ 3x^2 & 2x + 2 \end{bmatrix}.$$

   So the Wronskian is

$$\begin{aligned}W(x^3, x^2 + x) &= \begin{vmatrix} x^3 & x^2 + 2x \\ 3x^2 & 2x + 2 \end{vmatrix} \\ &= (x^3)(2x + 2) - (3x^2)(x^2 + 2x) \\ &= 2x^4 + 2x^3 - 3x^4 - 6x^3 \\ &= -x^4 - 4x^3,\end{aligned}$$

   which is nonzero except $x = -4$ and $x = 0$. So we conclude that the set $\{x^3, x^2 + 2x\}$ is linearly independent. □

   (2) $\{e^{-x}, e^x, e^{2x}\}$

   Solution. The Wronskian matrix is

$$\begin{bmatrix} e^{-x} & e^x & e^{2x} \\ \frac{d}{dx}e^{-x} & \frac{d}{dx}e^x & \frac{d}{dx}e^{2x} \\ \frac{d^2}{dx^2}e^{-x} & \frac{d^2}{dx^2}e^x & \frac{d^2}{dx^2}e^{2x} \end{bmatrix} = \begin{bmatrix} e^{-x} & e^x & e^{2x} \\ -e^{-x} & e^x & 2e^{2x} \\ e^{-x} & e^x & 4e^{2x} \end{bmatrix}.$$

   So the Wronskian is

$$\begin{aligned}W(x^3, x^2 + x) &= \begin{vmatrix} e^{-x} & e^x & e^{2x} \\ -e^{-x} & e^x & 2e^{2x} \\ e^{-x} & e^x & 4e^{2x} \end{vmatrix} \\ &= e^{-x}\begin{vmatrix} e^x & 2e^{2x} \\ e^x & 4e^{2x} \end{vmatrix} - e^x \begin{vmatrix} -e^{-x} & 2e^{2x} \\ e^{-x} & 4e^{2x} \end{vmatrix} + e^{2x}\begin{vmatrix} -e^{-x} & e^x \\ e^{-x} & e^x \end{vmatrix} \\ &= e^{-x}((e^x)(4e^{2x}) - (e^x)(2e^{2x})) - e^x((-e^{-x})(4e^{2x}) - (e^{-x})(2e^{2x})) \\ &\quad + e^{2x}((-e^{-x})(e^x) - (e^{-x})(e^x)) \\ &= e^{-x}(2e^{3x}) - e^x(-6e^x) + e^{2x}(-2) \\ &= 2e^{2x} + 6e^{2x} - 2e^{2x} \\ &= 6e^{2x},\end{aligned}$$

   which is nonzero for all real numbers x. So we conclude that the set $\{e^{-x}, e^x, e^{2x}\}$ is linearly independent. □

5. Find the general solution of

$$y''' - y'' - y' + y = 5,$$

   given that $y = 5$ is a particular solution.

Solution. The general solution for the ordinary differential equation is $y = y_h + y_p = y_h + 5$, where $y_p = 5$ is a particular solution and y_h is a homogeneous solution of

$$y_h''' - y_h'' - y_h' + y_h = 0.$$

Let $y = e^{\lambda x}$, where λ is a number. Then we obtain the first, second, and third derivatives

$$\begin{aligned} y_h' &= \lambda e^{\lambda x}, \\ y_h'' &= \lambda^2 e^{\lambda x}, \\ y_h''' &= \lambda^3 e^{\lambda x}. \end{aligned}$$

So we have

$$\begin{aligned} 0 &= y_h''' - y_h'' - y_h' + y_h \\ &= \lambda^3 e^{\lambda x} - \lambda^2 e^{\lambda x} - \lambda e^{\lambda x} + e^{\lambda x} \\ &= e^{\lambda x}(\lambda^3 - \lambda^2 - \lambda + 1) \\ &= e^{\lambda x}(\lambda + 1)(\lambda^2 - 2\lambda + 1) \\ &= e^{\lambda x}(\lambda + 1)(\lambda - 1)^2. \end{aligned}$$

Since we know $e^{\lambda x} \neq 0$, we must conclude $(\lambda + 1)(\lambda - 1)^2. = 0$. which gives the real root $\lambda_1 = -1$ and the repeated real roots $\lambda_2 = 1$. So the general solution is

$$\begin{aligned} y_h &= C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x} + C_3 x e^{\lambda_2 x} \\ &= C_1 e^{-x} + C_2 e^{x} + C_3 x e^{x}, \end{aligned}$$

and so the general solution is

$$\begin{aligned} y &= y_h(x) + y_p(x) \\ &= \boxed{C_1 e^{-x} + C_2 e^{x} + C_3 x e^{x} + 5}, \end{aligned}$$

where C_1, C_2, C_3, C_4 are constants. □