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Homework 6 solutions

1. Find the general solution of

$$y'' - 6y' + 10y = 0.$$

Solution. Let $y = e^{\lambda x}$, where λ is a number. Then we obtain the first and second derivatives

$$\begin{aligned} y' &= \lambda e^{\lambda x}, \\ y'' &= \lambda^2 e^{\lambda x}. \end{aligned}$$

So we have

$$\begin{aligned} 0 &= y'' - 6y' + 10y \\ &= \lambda^2 e^{\lambda x} - 6\lambda e^{\lambda x} + 10 e^{\lambda x} \\ &= e^{\lambda x}(\lambda^2 - 6\lambda + 10). \end{aligned}$$

Since we know $e^{\lambda x} \neq 0$, we must conclude $\lambda^2 - 6\lambda + 10 = 0$, which gives the imaginary roots $\lambda_1 = 3 - i$ and $\lambda_2 = 3 + i$. So the general solution is

$$\begin{aligned} y &= C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x} \\ &= C_1 e^{(3-i)x} + C_2 e^{(3+i)x} \\ &= e^{3x}(C_1 e^{ix} + C_2 e^{-ix}) \\ &= \boxed{e^{3x}(C_1 \cos(x) + C_2 \sin(x))}, \end{aligned}$$

where C_1, C_2 are constants. □

2. Given that $y_1 = x^2$ is a solution of

$$x^2 y'' + xy' - 4y = 0,$$

find the general solution using reduction of order.

Solution. Write $y = x^2 u$, where $u = u(x)$. Then we obtain the first and second derivatives

$$\begin{aligned} y' &= 2xu + x^2 u', \\ y'' &= 2u + 4xu' + x^2 u''. \end{aligned}$$

So we obtain

$$\begin{aligned} 0 &= x^2 y'' + xy' - 4y \\ &= x^2(2u + 4xu' + x^2 u'') + x(2xu + x^2 u') - 4x^2 u \\ &= 2x^2 u + 4x^3 u' + x^4 u'' + 2x^2 u + x^3 u' - 4x^2 u \\ &= x^4 u'' + 5x^3 u' \\ &= x^3(xu'' + 5u'), \end{aligned}$$

which implies either $x = 0$ or $xu'' + 5u' = 0$. If we have $x = 0$, then $y = x^2 u = 0^2 u = 0$, which is a trivial solution. But we are interested only in a nontrivial solution, which means we should assume

$$xu'' + 5u' = 0.$$

Let $w = u'$. Then we obtain

$$xw' + 5w = 0,$$

which is a separable first-order equation in w. We can rewrite the separable equation as

$$\frac{1}{w}\,dw = -\frac{5}{x}\,dx,$$

and we can integrate both sides of the separable equation, writing

$$\int \frac{1}{w}\,dw = \int -\frac{5}{x}\,dx,$$

in order to obtain

$$\ln(|w|) = -5\ln(|x|) + \ln(C_0),$$

or equivalently the solution

$$u' = w = \frac{C_0}{x^5},$$

where C_0 is an arbitrary constant. We can rewrite this separable equation as

$$du = \frac{C_0}{x^5}\,dx,$$

and we can integrate both sides of the separable equation, writing

$$\int 1\,du = \int \frac{C_0}{x^5}\,dx,$$

where C_0 is an arbitrary constant, in order to obtain

$$\frac{y}{x^2} = u = \frac{C_1}{x^4} + C_2,$$

or equivalently

$$\begin{aligned} y &= x^2\left(\frac{C_1}{x^4} + C_2\right) \\ &= \boxed{\frac{C_1}{x^2} + C_2x^2}, \end{aligned}$$

where C_1, C_2 are arbitrary constants. □

3. Find the solution of the initial value problem

$$\begin{aligned} y'' - 4y' + 5y &= 0, \\ y(0) &= 3, \\ y'(0) &= 1. \end{aligned}$$

Solution. Let $y = e^{\lambda x}$, where λ is a number. Then we obtain the first and second derivatives

$$\begin{aligned} y' &= \lambda e^{\lambda x}, \\ y'' &= \lambda^2 e^{\lambda x}. \end{aligned}$$

So we have

$$\begin{aligned} 0 &= y'' - 4y' + 5y \\ &= \lambda^2 e^{\lambda x} - 4\lambda e^{\lambda x} + 5e^{\lambda x} \\ &= e^{\lambda x}(\lambda^2 - 4\lambda + 5). \end{aligned}$$

Since we know $e^{\lambda x} \neq 0$, we must conclude $\lambda^2 - 4\lambda + 5 = 0$, which gives the imaginary roots $\lambda_1 = 2 + i$ and $\lambda_2 = 2 - i$. So the general solution is

$$\begin{aligned} y &= C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x} \\ &= C_1 e^{(2+i)x} + C_2 e^{(2-i)x} \\ &= e^{2x}(C_1 e^{ix} + C_2 e^{-ix}) \\ &= e^{2x}(C_1\cos(x) + C_2\sin(x)), \end{aligned}$$

where C_1, C_2 are constants. We also obtain the derivative

$$\begin{aligned} y' &= 2e^{2x}(C_1\cos(x) + C_2\sin(x)) + e^{2x}(-C_1\sin(x) + C_2\cos(x)) \\ &= e^{2x}((2C_1 + C_2)\cos(x) + (2C_2 - C_1)\sin(x)). \end{aligned}$$

Now, we can apply the initial condition $y(0) = 3$ and $y'(0) = 1$ to obtain the linear system of equations

$$\begin{aligned} 3 &= C_1, \\ 1 &= 2C_1 + C_2, \end{aligned}$$

from which we can solve simultaneously to deduce $C_1 = 3$ and $C_2 = -5$. Therefore,

$$\begin{aligned} y &= e^{2x}(C_1\cos(x) + C_2\sin(x)) \\ &= \boxed{e^{2x}(3\cos(x) - 5\sin(x))}, \end{aligned}$$

is the solution to the initial value problem. □

4. Find the solution of
$$y'' - 5y' + 6y = e^{3x}.$$

Solution. First, we will find the homogeneous solution y_h, which solves
$$y_h'' - 5y_h' + 6y_h = 0.$$
Let $y_h = e^{\lambda x}$, where λ is a number. Then we obtain the first and second derivatives
$$\begin{aligned} y_h' &= \lambda e^{\lambda x}, \\ y_h'' &= \lambda^2 e^{\lambda x}. \end{aligned}$$
So we have
$$\begin{aligned} 0 &= y_h'' - 5y_h' + 6y_h \\ &= \lambda^2 e^{\lambda x} - 5\lambda e^{\lambda x} + 6e^{\lambda x} \\ &= e^{\lambda x}(\lambda^2 - 5\lambda + 6) \\ &= e^{\lambda x}(\lambda - 2)(\lambda - 3). \end{aligned}$$
Since we know $e^{\lambda x} \neq 0$, we must conclude $(\lambda - 2)(\lambda - 3) = 0$, which gives the distinct real roots $\lambda_1 = 2$ and $\lambda_2 = 3$. So the homogeneous solution is
$$\begin{aligned} y_h &= C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x} \\ &= C_1 e^{2x} + C_2 e^{3x}, \end{aligned}$$
where C_1, C_2 are constants. Now, we will find the particular solution y_p, which solves
$$y_p'' - 5y_p' + 6y_p = e^{3x},$$
using the method of undetermined coefficients. We cannot assume the form $y_p = Ae^{3x}$ because that would make y_p a linear combination of y_h. Instead, the particular solution takes the form $y_p = Axe^{3x}$, where A is a constant. We obtain the derivatives
$$\begin{aligned} y_p' &= Ae^{3x}(3x + 1), \\ y_p'' &= 3Ae^{3x}(3x + 2). \end{aligned}$$
So we have
$$\begin{aligned} e^{3x} &= y_p'' - 5y_p' + 6y_p \\ &= 3Ae^{3x}(3x + 2) - 5(Ae^{3x}(3x + 1)) + 6Axe^{3x} \\ &= 9Axe^{3x} + 6Ae^{3x} - 15Axe^{3x} - 5Ae^{3x} + 6Axe^{3x} \\ &= Ae^{3x}, \end{aligned}$$
from which we deduce $A = 1$, and so our particular solution is
$$\begin{aligned} y_p &= Axe^{3x} \\ &= xe^{3x}. \end{aligned}$$
Therefore,
$$\begin{aligned} y &= y_h + y_p \\ &= \boxed{C_1 e^{2x} + C_2 e^{3x} + xe^{3x}}, \end{aligned}$$
is the general solution to the problem. □

5. Find the solution of
$$y'' + y' + 2y = x^2 + 4.$$

Solution. First, we will find the homogeneous solution y_h, which solves
$$y_h'' + y_h' + 2y_h = 0.$$
Let $y_h = e^{\lambda x}$, where λ is a number. Then we obtain the first and second derivatives
$$\begin{aligned} y_h' &= \lambda e^{\lambda x}, \\ y_h'' &= \lambda^2 e^{\lambda x}. \end{aligned}$$

So we have

$$\begin{aligned}0 &= y_h'' + y_h' + 2y_h \\ &= \lambda^2 e^{\lambda x} + \lambda e^{\lambda x} + 2e^{\lambda x} \\ &= e^{\lambda x}(\lambda^2 + \lambda + 2).\end{aligned}$$

Since we know $e^{\lambda x} \neq 0$, we must conclude $\lambda^2 + \lambda + 2 = 0$, which gives the imaginary roots $\lambda_1 = -\frac{1}{2} + \frac{\sqrt{7}}{2}i$ and $\lambda_2 = -\frac{1}{2} - \frac{\sqrt{7}}{2}i$. So the homogeneous solution is

$$\begin{aligned}y_h &= C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x} \\ &= C_1 e^{(-\frac{1}{2} + \frac{\sqrt{7}}{2}i)x} + C_2 e^{(-\frac{1}{2} - \frac{\sqrt{7}}{2}i)x} \\ &= e^{-\frac{1}{2}x}(C_1 e^{\frac{\sqrt{7}}{2}ix} + C_2 e^{-\frac{\sqrt{7}}{2}ix}) \\ &= e^{-\frac{1}{2}x}\left(C_1 \cos\left(\frac{\sqrt{7}}{2}x\right) + C_2 \sin\left(\frac{\sqrt{7}}{2}x\right)\right),\end{aligned}$$

where C_1, C_2 are constants. Now, we will find the particular solution y_p, which solves

$$y_p'' + y_p' + 2y_p = x^2 + 4,$$

using the method of undetermined coefficients. The particular solution takes the form $y_p = Ax^2 + Bx + C$, where A, B, C are constants. We obtain the derivatives

$$\begin{aligned}y_p' &= 2Ax + B, \\ y_p'' &= 2A.\end{aligned}$$

So we have

$$\begin{aligned}x^2 + 4 &= y_p'' + y_p' + 2y_p \\ &= 2A + (2Ax + B) + 2(Ax^2 + Bx + C) \\ &= 2A + 2Ax + B + 2Ax^2 + 2Bx + 2C \\ &= 2Ax^2 + (2A + 2B)x + (2A + B + 2C),\end{aligned}$$

from which we can equate the coefficients to obtain a linear system of equations

$$\begin{aligned}2A &= 1, \\ 2A + 2B &= 0, \\ 2A + B + 2C &= 4.\end{aligned}$$

We can solve simultaneously this linear system to deduce $A = \frac{1}{2}$, $B = -\frac{1}{2}$, $C = \frac{7}{4}$. So our particular solution is

$$\begin{aligned}y_p &= Ax^2 + Bx + C \\ &= \frac{1}{2}x^2 - \frac{1}{2}x + \frac{7}{4}.\end{aligned}$$

Therefore,

$$\begin{aligned}y &= y_h + y_p \\ &= \boxed{e^{-\frac{1}{2}x}\left(C_1 \cos\left(\frac{\sqrt{7}}{2}x\right) + C_2 \sin\left(\frac{\sqrt{7}}{2}x\right)\right) + \frac{1}{2}x^2 - \frac{1}{2}x + \frac{7}{4}},\end{aligned}$$

is the general solution to the problem. □

6. Find the solution of

$$y'' + y' = \sin(x).$$

Solution. First, we will find the homogeneous solution y_h, which solves

$$y_h'' + y_h' = 0.$$

Let $z = y_h'$. Then we obtain

$$z' + z = 0,$$

which is a separable first-order equation in z. We can rewrite the separable equation as

$$-\frac{1}{z}\,dz = dx,$$

and we can integrate both sides of the separable equation, writing

$$\int -\frac{1}{z}\,dz = \int 1\,dx,$$

in order to obtain

$$-\ln(|z|) = x + C_0,$$

or equivalently the solution

$$y_h' = z = e^{-C_0}e^{-x},$$

where C_0 is an arbitrary constant. We can rewrite this separable equation as

$$dy_h = e^{-C_0}e^{-x}\,dx,$$

and we can integrate both sides of the separable equation, writing

$$\int 1\,dy_h = \int e^{-C_0}e^{-x}\,dx,$$

in order to obtain the homogeneous solution

$$y_h = C_1e^{-x} + C_2,$$

where C_1, C_2 are arbitrary constants. Now, we will find the particular solution y_p, which solves

$$y_p'' + y_p' = \sin(x),$$

using the method of undetermined coefficients. The particular solution takes the form $y_p = A\cos(x) + B\sin(x)$, where A, B are constants. We obtain the derivatives

$$\begin{aligned} y_p' &= -A\sin(x) + B\cos(x), \\ y_p'' &= -A\cos(x) - B\sin(x). \end{aligned}$$

So we have

$$\begin{aligned} \sin(x) &= y_p'' + y_p' \\ &= (-A\cos(x) - B\sin(x)) + (-A\sin(x) + B\cos(x)) \\ &= (-A + B)\cos(x) + (-A - B)\sin(x), \end{aligned}$$

from which we can equate the coefficients to obtain a linear system of equations

$$\begin{aligned} -A + B &= 0, \\ -A - B &= 1. \end{aligned}$$

We can solve simultaneously this linear system to deduce $A = -\frac{1}{2}$ and $B = -\frac{1}{2}$. So our particular solution is

$$\begin{aligned} y_p &= A\cos(x) + B\sin(x) \\ &= -\frac{1}{2}\cos(x) - \frac{1}{2}\sin(x). \end{aligned}$$

Therefore,

$$\begin{aligned} y &= y_h + y_p \\ &= \boxed{C_1e^{-x} + C_2 - \frac{1}{2}\cos(x) - \frac{1}{2}\sin(x)}, \end{aligned}$$

is the general solution to the problem. □