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Homework 7 solutions

1. Find the general solution of

$$y'' + y = \sec(x).$$

Solution. First, we will find the homogeneous solution y_h, which solves

$$y_h'' + y_h = 0.$$

Let $y_h = e^{\lambda x}$, where λ is a number. Then we obtain the first and second derivatives

$$\begin{aligned} y_h' &= \lambda e^{\lambda x}, \\ y_h'' &= \lambda^2 e^{\lambda x}. \end{aligned}$$

So we have

$$\begin{aligned} 0 &= y_h'' + y_h \\ &= \lambda^2 e^{\lambda x} + e^{\lambda x} \\ &= e^{\lambda x}(\lambda^2 + 1). \end{aligned}$$

Since we know $e^{\lambda x} \neq 0$, we must conclude $\lambda^2 + 1 = 0$, which gives the imaginary roots $\lambda_1 = i$ and $\lambda_2 = i$. So the homogeneous solution is

$$\begin{aligned} y_h &= C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x} \\ &= C_1 e^{ix} + C_2 e^{-ix} \\ &= C_1 \cos(x) + C_2 \sin(x), \end{aligned}$$

where C_1, C_2 are constants. Next, we will find the particular solution y_p, which solves

$$y_p'' + y_p = \sec(x),$$

using variation of parameters. We compute the Wronskian matrix

$$\begin{bmatrix} \cos(x) & \sin(x) \\ \frac{d}{dx}\cos(x) & \frac{d}{dx}\sin(x) \end{bmatrix} = \begin{bmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{bmatrix}$$

and hence the Wronksian

$$\begin{aligned} W(\cos(x), \sin(x)) &= \begin{vmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{vmatrix} \\ &= (\cos(x))(\cos(x)) - (-\sin(x))(\sin(x)) \\ &= \cos^2(x) + \sin^2(x) \\ &= 1. \end{aligned}$$

We also obtain

$$\begin{aligned} W_1 &= \begin{vmatrix} 0 & \sin(x) \\ \sec(x) & \cos(x) \end{vmatrix} \\ &= (0)(\cos(x)) - (\sec(x))(\sin(x)) \\ &= -\frac{\sin(x)}{\cos(x)} \\ &= -\tan(x) \end{aligned}$$

and

$$\begin{aligned} W_2 &= \begin{vmatrix} \cos(x) & 0 \\ -\sin(x) & \sec(x) \end{vmatrix} \\ &= (\cos(x))(\sec(x)) - (-\sin(x))(0) \\ &= \frac{\cos(x)}{\cos(x)} \\ &= 1. \end{aligned}$$

So we have

$$v_1' = \frac{W_1}{W} = \frac{-\tan(x)}{1} = -\tan(x),$$
$$v_2' = \frac{W_2}{W} = \frac{1}{1} = 1,$$

which imply

$$v_1 = \int v_1'\,dx = \int -\tan(x)\,dx = \ln(|\cos(x)|),$$
$$v_2 = \int v_2'\,dx = \int 1\,dx = x,$$

respectively. So our particular solution is

$$\begin{aligned} y_p &= v_1\cos(x) + v_2\sin(x) \\ &= \ln(|\cos(x)|)\cos(x) + x\sin(x). \end{aligned}$$

Therefore,

$$\begin{aligned} y &= y_h + y_p \\ &= \boxed{C_1\cos(x) + C_2\sin(x) + \ln(|\cos(x)|)\cos(x) + x\sin(x)} \end{aligned}$$

is the general solution to the problem. □

2. Find the general solution of

$$(x^2-1)y'' - 2xy' + 2y = (x^2-1)^2$$

if two solutions to the associated homogeneous equation are x and x^2+1.

Solution. First, we rewrite the ordinary differential equation in its standard form

$$y'' - \frac{2x}{x^2-1}y' + \frac{2}{x^2-1}y = x^2-1.$$

As we are given in the problem statement that the two solutions to the associated homogeneous equation are x and x^2+1, our homogeneous solution is

$$y_h = C_1x + C_2(x^2+1),$$

where C_1, C_2 are arbitrary constants. Next, we will find the particular solution y_p, which solves

$$y_p'' - \frac{2x}{x^2-1}y_p' + \frac{2}{x^2-1}y_p = x^2-1,$$

using variation of parameters. We compute the Wronskian matrix

$$\begin{bmatrix} x & x^2+1 \\ \frac{d}{dx}x & \frac{d}{dx}(x^2+1) \end{bmatrix} = \begin{bmatrix} x & x^2-1 \\ 1 & 2x \end{bmatrix}$$

and hence the Wronksian

$$\begin{aligned} W(x, x^2+1) &= \begin{vmatrix} x & x^2+1 \\ 1 & 2x \end{vmatrix} \\ &= (x)(2x) - (1)(x^2+1) \\ &= x^2-1. \end{aligned}$$

We also obtain

$$\begin{aligned} W_1 &= \begin{vmatrix} 0 & x^2+1 \\ x^2-1 & 2x \end{vmatrix} \\ &= (0)(2x) - (x^2-1)(x^2+1) \\ &= -(x^2-1)(x^2+1) \end{aligned}$$

and

$$\begin{aligned} W_2 &= \begin{vmatrix} x & 0 \\ 1 & x^2-1 \end{vmatrix} \\ &= (x)(x^2-1) - (1)(0) \\ &= x(x^2-1). \end{aligned}$$

So we have

$$\begin{aligned} v_1' &= \frac{W_1}{W} = \frac{-(x^2-1)(x^2+1)}{x^2-1} = -x^2 - 1, \\ v_2' &= \frac{W_2}{W} = \frac{x(x^2-1)}{x^2-1} = x, \end{aligned}$$

which imply

$$\begin{aligned} v_1 &= \int v_1'\,dx = \int -x^2 - 1\,dx = -\frac{1}{3}x^3 - x, \\ v_2 &= \int v_2'\,dx = \int x\,dx = \frac{1}{2}x^2, \end{aligned}$$

respectively. So our particular solution is

$$\begin{aligned} y_p &= v_1 x + v_2(x^2+1) \\ &= \left(-\frac{1}{3}x^3 - x\right)x + \frac{1}{2}x^2(x^2+1) \\ &= \frac{1}{6}x^4 - \frac{1}{2}x^2. \end{aligned}$$

Therefore,

$$\begin{aligned} y &= y_h + y_p \\ &= \boxed{C_1 x + C_2(x^2+1) + \frac{1}{6}x^4 - \frac{1}{2}x^2} \end{aligned}$$

is the general solution to the problem. □

3. Solve the initial value problem

$$\begin{aligned} y'' + y &= x, \\ y(1) &= 0, \\ y'(1) &= 1. \end{aligned}$$

Solution. We have already established in our solution to Exercise 1 that the homogeneous solution is

$$y_h = C_1\cos(x) + C_2\sin(x),$$

where C_1, C_2 are constants. We will find the particular solution y_p, which solves

$$y_p'' + y_p = x,$$

using the method of undetermined coefficients. The particular solution takes the form $y_p = Ax + B$, where A, B are constants. We obtain the derivatives

$$\begin{aligned} y_p' &= A, \\ y_p'' &= 0. \end{aligned}$$

So we have

$$\begin{aligned} x &= y_p'' + y_p \\ &= 0 + (Ax + B) \\ &= Ax + B, \end{aligned}$$

from which we can equate the coefficients to deduce $A = 1$ and $B = -0$. So our particular solution is

$$\begin{aligned} y_p &= Ax + B \\ &= 1x + 0 \\ &= x. \end{aligned}$$

Therefore, our general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= C_1\cos(x) + C_2\sin(x) + x. \end{aligned}$$

Next, we consider the initial conditions $y(1) = 0$ and $y'(1) = 1$. We recall

$$\begin{aligned} y &= C_1 \cos(x) + C_2 \sin(x) + x, \\ y' &= -C_1 \sin(x) + C_2 \cos(x) + 1. \end{aligned}$$

At $x = 1$, this becomes

$$\begin{aligned} 0 &= C_1 \cos(1) + C_2 \sin(1) + 1, \\ 1 &= -C_1 \sin(1) + C_2 \cos(1) + 1, \end{aligned}$$

from which we can solve simultaneously to obtain $C_1 = -\cos(1)$ and $C_2 = -\sin(1)$. Therefore,

$$\begin{aligned} y &= C_1 \cos(x) + C_2 \sin(x) + x \\ &= -\cos(1)\cos(x) - \sin(1)\sin(x) + x \\ &= -(\cos(1)\cos(x) + \sin(1)\sin(x)) + x \\ &= -(\cos(-1)\cos(x) - \sin(-1)\sin(x)) + x \\ &= -\cos(-1 + x) + x \\ &= \boxed{x - \cos(x - 1)} \end{aligned}$$

is the solution to the initial value problem. □

4. A 20 lb weight is suspended from the end of a vertical spring having a spring constant of 40 lb/ft and is allowed to reach equilibrium. It is then set into motion by stretching the spring 2 in from its equilibrium position and releasing the mass from rest. Find the position of the weight at any time t if there is no external force and no air resistance.

Solution. The differential equation of free undamped motion is

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0,$$

where, m is the mass and k is the spring constant. We are given that the spring constant of a vertical spring is $k = 40$ lb/ft. The mass can be found from the equation $W = mg$, where $W = 20$ lb is the weight and $g = 32$ ft/s^2 is the acceleration of gravity; we find

$$\begin{aligned} m &= \frac{W}{g} \\ &= \frac{20 \text{ lb}}{32 \text{ ft/s}^2} \\ &= \frac{5}{8} \text{ slug}. \end{aligned}$$

So our differential equation is

$$\frac{d^2x}{dt^2} + \frac{40}{\frac{5}{8}}x = 0,$$

or

$$\frac{d^2x}{dt^2} + 64x = 0.$$

Finally, by pulling a mass on a spring down 2 in $= \frac{1}{6}$ ft below the equilibrium position, holding it until $t = 0$, and then releasing it from rest, we obtain the initial conditions $x(0) = \frac{1}{6}$ and $x'(0) = 0$. So we have formulated the initial value problem

$$\begin{aligned} x'' + 64x &= 0, \\ x(0) &= \frac{1}{6}, \\ x'(0) &= 0. \end{aligned}$$

Now, we will now need to solve the initial value problem. Let $x = e^{\lambda t}$, where λ is a number. Then we obtain the first and second derivatives

$$\begin{aligned} x' &= \lambda e^{\lambda t}, \\ x'' &= \lambda^2 e^{\lambda t}. \end{aligned}$$

So we have

$$\begin{aligned} 0 &= x'' + 64x \\ &= \lambda^2 e^{\lambda t} + 64 e^{\lambda t} \\ &= e^{\lambda t}(\lambda^2 + 64). \end{aligned}$$

Since we know $e^{\lambda x} \neq 0$, we must conclude $\lambda^2 + 64 = 0$, which gives the imaginary roots $\lambda_1 = 8i$ and $\lambda_2 = -8i$. So the general solution is

$$\begin{aligned} x(t) &= C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} \\ &= C_1 e^{8it} + C_2 e^{-8it} \\ &= C_1 \cos(8t) + C_2 \sin(8t), \end{aligned}$$

where C_1, C_2 are constants. We also obtain the derivative

$$x'(t) = -C_1 \sin(8t) + C_2 \cos(8t).$$

Now, we can apply the initial condition $x(0) = \frac{1}{6}$ and $x'(0) = 0$ to deduce $C_1 = \frac{1}{6}$ and $C_2 = 0$. Therefore,

$$\begin{aligned} x(t) &= C_1 \cos(8t) + C_2 \sin(8t) \\ &= \frac{1}{6} \cos(8t) + 0 \sin(8t) \\ &= \boxed{\frac{1}{6} \cos(8t)}, \end{aligned}$$

is the solution to the initial value problem. □