MATH 046 discussion
University of California, Riverside

Ryan Ta
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Homework 8 solutions

1. Find the general solution of
$$4x^2y'' + 4xy' - y = 0.$$

Solution. Let $y = x^\lambda$, where λ is a number. Then we obtain the first and second derivatives
$$\begin{aligned} y' &= \lambda x^{\lambda-1}, \\ y'' &= \lambda(\lambda-1)x^{\lambda-2}. \end{aligned}$$

So we have
$$\begin{aligned} 0 &= 4x^2y'' + 4xy' - y \\ &= 4x^2(\lambda(\lambda-1)x^{\lambda-2}) + 4x(\lambda x^{\lambda-1}) - x^\lambda \\ &= x^\lambda(4\lambda^2 - 4\lambda) + x^\lambda(4\lambda) - x^\lambda \\ &= x^\lambda(4\lambda^2 - 1). \end{aligned}$$

If we assume $x^\lambda = 0$, then we would have $y = x^\lambda = 0$, meaning that the solution would be trivial. As we are interested in a nontrivial solution, we should assume $4\lambda^2 - 1 = 0$, which gives the distinct real roots $\lambda_1 = \frac{1}{2}$ and $\lambda_2 = -\frac{1}{2}$. So the general solution is
$$\begin{aligned} y &= C_1x^{\lambda_1} + C_2x^{\lambda_2} \\ &= C_1x^{\frac{1}{2}} + C_2x^{-\frac{1}{2}} \\ &= \boxed{C_1\sqrt{x} + \frac{C_2}{\sqrt{x}}}, \end{aligned}$$

where C_1, C_2 are constants. □

2. Find the general solution of
$$x^2y'' - 3xy' + 4y = 0.$$

Solution. Let $y = x^\lambda$, where λ is a number. Then we obtain the first and second derivatives
$$\begin{aligned} y' &= \lambda x^{\lambda-1}, \\ y'' &= \lambda(\lambda-1)x^{\lambda-2}. \end{aligned}$$

So we have
$$\begin{aligned} 0 &= x^2y'' - 3xy' + 4y \\ &= x^2(\lambda(\lambda-1)x^{\lambda-2}) - 3x(\lambda x^{\lambda-1}) + 4x^\lambda \\ &= x^\lambda(\lambda^2 - \lambda) - x^\lambda(3\lambda) + 4x^\lambda \\ &= x^\lambda(\lambda^2 - 4\lambda + 4) \\ &= x^\lambda(\lambda-2)^2. \end{aligned}$$

If we assume $x^\lambda = 0$, then we would have $y = x^\lambda = 0$, meaning that the solution would be trivial. As we are interested in a nontrivial solution, we should assume $(\lambda-2)^2 = 0$, which gives the repeated roots $\lambda_1 = 2$. So the general solution is
$$\begin{aligned} y &= C_1x^{\lambda_1} + C_2x^{\lambda_1}\ln(x) \\ &= \boxed{C_1x^2 + C_2x^2\ln(x)}, \end{aligned}$$

where C_1, C_2 are constants. □

3. Find the general solution of
$$2x^2y'' + 11xy' + 4y = 0.$$

Solution. Let $y = x^\lambda$, where λ is a number. Then we obtain the first and second derivatives

$$\begin{aligned} y' &= \lambda x^{\lambda-1}, \\ y'' &= \lambda(\lambda-1)x^{\lambda-2}. \end{aligned}$$

So we have

$$\begin{aligned} 0 &= 2x^2y'' + 11xy' + 4y \\ &= 2x^2(\lambda(\lambda-1)x^{\lambda-2}) + 11x(\lambda x^{\lambda-1}) + 4x^\lambda \\ &= x^\lambda(2\lambda^2 - 2\lambda) + x^\lambda(11\lambda) + 4x^\lambda \\ &= x^\lambda(2\lambda^2 + 9\lambda + 4) \\ &= x^\lambda(2x+1)(x+4). \end{aligned}$$

If we assume $x^\lambda = 0$, then we would have $y = x^\lambda = 0$, meaning that the solution would be trivial. As we are interested in a nontrivial solution, we should assume $(2x+1)(x+4) = 0$, which gives the distinct real roots $\lambda_1 = -\frac{1}{2}$ and $\lambda_2 = -4$. So the general solution is

$$\begin{aligned} y &= C_1x^{\lambda_1} + C_2x^{\lambda_2} \\ &= C_1x^{-\frac{1}{2}} + C_2x^{-4} \\ &= \boxed{\frac{C_1}{\sqrt{x}} + \frac{C_2}{x^4}}, \end{aligned}$$

where C_1, C_2 are constants. □

4. Find the general solution of

$$x^2y'' - 2y = 0.$$

Solution. Let $y = x^\lambda$, where λ is a number. Then we obtain the first and second derivatives

$$\begin{aligned} y' &= \lambda x^{\lambda-1}, \\ y'' &= \lambda(\lambda-1)x^{\lambda-2}. \end{aligned}$$

So we have

$$\begin{aligned} 0 &= x^2y'' - 2y \\ &= x^2(\lambda(\lambda-1)x^{\lambda-2}) - 2x^\lambda \\ &= x^\lambda(\lambda^2 - \lambda) - 2x^\lambda \\ &= x^\lambda(\lambda^2 - \lambda - 2) \\ &= x^\lambda(\lambda-2)(\lambda+1). \end{aligned}$$

If we assume $x^\lambda = 0$, then we would have $y = x^\lambda = 0$, meaning that the solution would be trivial. As we are interested in a nontrivial solution, we should assume $(\lambda-2)(\lambda+1) = 0$, which gives the distinct real roots $\lambda_1 = 2$ and $\lambda_2 = -1$. So the general solution is

$$\begin{aligned} y &= C_1x^{\lambda_1} + C_2x^{\lambda_2} \\ &= C_1x^2 + C_2x^{-1} \\ &= \boxed{C_1x^2 + \frac{C_2}{x}}, \end{aligned}$$

where C_1, C_2 are constants. □

5. Find the general solution of

$$x^2y'' - 6xy' = 0.$$

Solution. Let $y = x^\lambda$, where λ is a number. Then we obtain the first and second derivatives

$$\begin{aligned} y' &= \lambda x^{\lambda-1}, \\ y'' &= \lambda(\lambda-1)x^{\lambda-2}. \end{aligned}$$

So we have

$$\begin{aligned}
0 &= x^2 y'' - 6xy' \\
&= x^2(\lambda(\lambda-1)x^{\lambda-2}) - 6x(\lambda x^{\lambda-1}) \\
&= x^\lambda(\lambda^2 - \lambda) - 6\lambda x^\lambda \\
&= x^\lambda(\lambda^2 - \lambda - 6\lambda) \\
&= x^\lambda(\lambda^2 - 7\lambda) \\
&= x^\lambda \lambda(\lambda - 7).
\end{aligned}$$

If we assume $x^\lambda = 0$, then we would have $y = x^\lambda = 0$, meaning that the solution would be trivial. As we are interested in a nontrivial solution, we should assume $\lambda(\lambda-7) = 0$, which gives the distinct real roots roots $\lambda_1 = 7$ and $\lambda_2 = 0$. So the general solution is

$$\begin{aligned}
y &= C_1 x^{\lambda_1} + C_2 x^{\lambda_2} \\
&= C_1 x^7 + C_2 x^0 \\
&= \boxed{C_1 x^7 + C_2},
\end{aligned}$$

where C_1, C_2 are constants. $\square$