7/10 Discussion

Proof by contradiction

Example 1: Prove that, if n^2 is even, then n is even.

Proof: Suppose by contradiction that n is odd. Then there exist an integer k such that $n = 2k+1$. So we have

$$n^2 = (2k+1)^2$$
$$= 4k^2 + 4k + 1$$
$$= 2(2k^2 + 2k) + 1$$

Since $2k^2 + 2k$ is also an integer, we conclude that n^2 is odd.

But this contradicts our assumption in the premises that n^2 is even. So this completes our proof by contradiction, and we conclude that n is even.

Example 2: Prove that $\sqrt{2}$ is irrational.

Proof: Assume by contradiction that $\sqrt{2}$ is rational. Then there exist integers a, b - with no common factors and $b \neq 0$ such that

$$\sqrt{2} = \frac{a}{b}$$

From $\sqrt{2} = \frac{a}{b}$, we obtain

$$2 = \frac{a^2}{b^2}$$

or equivalently

$$a^2 = 2b^2$$

Since b^2 is an integer, it follows that a^2 is even.

We claim that: if a^2 is even, then a is even.

To prove our claim, suppose by contradiction that a is odd. Then there exists an integer k such that

$$a = 2k+1.$$

So we have

$$\begin{aligned} a^2 &= (2k+1)^2 \\ &= 4k^2+4k+1 \\ &= 2(2k^2+2k)+1. \end{aligned}$$

Since $2k^2+2k$ is also an integer, it follows that a^2 is odd. But this contradicts our assumption that a^2 is even. Therefore, a is even, satisfying our claim.

By the claim, we have that a is even.
Now we return to the equation

$$a^2 = 2b^2.$$

Since a is even, there exists an integer l such that

$$a = 2l.$$

So we obtain

$$(2l)^2 = 2b^2$$
$$4l^2 = 2b^2$$
$$2l^2 = b^2$$
$$(b^2 = 2l^2)$$

So b^2 is even.

Since b^2 is even, by the claim we made earlier, we have that b is even.

Therefore, a and b are both even. In other words, 2 is a common factor of a and b. But this contradicts ~~our~~ ~~the earlier~~ assumption earlier in this proof that our a and b have no common factors (greater than 1)

So we conclude that $\sqrt{2}$ is irrational. ∎

$\sqrt{2} = \frac{a}{b}$ a, b are integers such that
- $b \neq 0$
- a and b have no common factor greater than 1.