DISCUSSION

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Proof by contradiction

Example 1: Prove that, if n^2 is even, then n is even

Proof: Suppose by contradiction that n is odd.
Then there exist an integer k such that $n = 2k+1$

So we have

$$n^2 = (2k+1)^2$$
$$= 4k^2 + 4k + 1$$
$$= 2(2k^2+2k) + 1$$

Since $2k^2+2k$ is also an integer, we conclude that n^2 is odd

But this contradicts our assumption in the premises that n^2 is even. So this completes our proof by contradiction, and we conclude that n is even.

Proof by contradiction

Example 2: Prove that $\sqrt{2}$ is irrational

Proof: Assume by contradiction that $\sqrt{2}$ is rational.
Then there exist integers a, b with no common factors and $b \neq 0$ - such that

$$\sqrt{2} = \frac{a}{b}$$

From $\sqrt{2} = \frac{a}{b}$, we obtain

$$2 = \frac{a^2}{b^2}$$

or equivalently,

$$a^2 = 2b^2$$

Since b^2 is an integer, it follows that a^2 is even

[we claim that: if a^2 is even, then a is even.] claim

To prove our claim, suppose by contradiction that a is odd
Then there exist an integer k such that

$$a = 2k+1$$

So we have

$$a^2 = (2k+1)^2$$
$$= 4k^2 + 4k + 1$$
$$= 2(2k^2+2k)+1$$

Since $2k^2+2k$ is also an integer, it follows that a^2 is odd

But this contradicts our assumption that a^2 is even.

Therefore, a is even, satisfying our claim

By the claim,

we have that a is even.

Now we return to the equation

$$a^2 = 2b^2$$

Since a is even, there exists an integer ℓ such that

$$a = 2\ell$$

So we obtain

$$(2\ell)^2 = 2b^2$$
$$4\ell^2 = 2b^2$$
$$2\ell^2 = b^2$$
$$(b^2 = 2\ell^2)$$

So b^2 is even

Since b^2 is even, by the claim we made earlier, we have that b is even

Therefore a and be are both even. In other words, 2 is a common factor of a and b. But this contradicts assumption earlier in this proof that a and b have no common factors (greater than 1)

So we conclude that $\sqrt{2}$ is irrational

$\sqrt{2} = \frac{a}{b}$

a, b are integers such that

$b \neq 0$

a and b have no common factor greater than 1