Discussion #3

Proof by contradiction

Example 1 Prove that, if n^2 is even, then n is even.

Proof: Suppose by contradiction that n is odd.
Then there exists an integer k such that

$$n = 2k+1$$

So we have
$$\begin{aligned} n^2 &= (2k+1)^2 \\ &= 4k^2+4k+1 \\ &= 2(2k^2+2k)+1 \end{aligned}$$

~~Since~~

Since $2k^2+2k$ is also an integer, we conclude that n^2 is odd. But this contradicts our assumption in the premises that n^2 is even. So this completes our proof by contradiction, and we conclude that n is even.

Example 2 : Prove that $\sqrt{2}$ is irrational.

Proof: Assume by contradiction that $\sqrt{2}$ is rational. Then there exist integers a, b — with no common factors and $b \neq 0$ — such that

$$\sqrt{2} = \frac{a}{b}.$$

$b \neq 0$, a and b have no common factors greater than 1.

From $\sqrt{2} = \frac{a}{b}$, we obtain $2 = \frac{a^2}{b^2}$,

or equivalently, $a^2 = 2b^2$.

a, b are integers such that
- $b \neq 0$
- a and b have no common factor greater than 1.

Since b^2 is an integer, it follows that a^2 is even.
We claim that: if a^2 is even, then a is even.
To prove our claim, suppose by contradiction that a is odd.
Then there exist an integer k such that

$$a = 2k+1.$$

So we'll have

$$a^2 = (2k+1)^2$$
$$= 4k^2+4k+1$$
$$= 2(2k^2+2k)+1$$

Since $2k^2+2k$ is also an integer, it follows that a^2 is odd. But this contradicts our assumption that a^2 is even. Thereforefore, a is even, satisfying our claim.

By the claim, we have that a is even. (our claim that: if a^2 is even, then a is even)

Now we return to the equation $a^2 = 2b^2$.

Since a is even, there exists, an integer l such that

$$a = 2l$$

So we obtain

$$(2l)^2 = 2b^2$$
$$4l^2 = 2b^2$$
$$2l^2 = b^2$$
$$(b^2 = 2l^2)$$

So b^2 is even.

Since b^2 is even, by the claim we made ealier, we have that b is even.

Therefore, a and b are both even. In other words, 2 is a common factor of a and b. But this contradicts our assumption earlier in this proof that a and b have no common factors (greater than 1).

So we conclude that $\sqrt{2}$ is irrational.