Discussion Note

Induction

Ex.1 Prove that we have $1+2+\ldots+n=\frac{n(n+1)}{2}$
for all positive integers n.

Proof: we will prove by induction.

Base step: For $n=1$, we have

$$1=\frac{(1)(1+1)}{2}$$

$$1=1$$

Induction step: For $n=k$, we assume

$$1+2+\ldots+k=\frac{k(k+1)}{2}$$

We will prove the statement for $n=k+1$, we have

$$1+2+\ldots+(k+1)=(1+\cdots+k)+(k+1)$$

$$=\frac{k(k+1)}{2}+(k+1)$$

$$=\frac{k(k+1)+2(k+1)}{2}$$

$$=\frac{(k+1)(k+2)}{2}$$

Ex. 2 Prove that we have

$1^2+\ldots+n^2=\frac{n(n+1)(2n+1)}{6}$ for all positive integers n.

Proof: For $n=1$ we have

Base step $\rightarrow$ $1^2=\frac{(1)(1+1)(1+2)}{6}$

$$1=1$$

Induction step: For $n=k$, we assume

$$1^2+\cdots+k^2=(k)(k+1)(2k+1)/6$$

For $n=k+1$, we have

$$1^2+\dots+(k+1)^2 = (1+\dots+k^2)+(k+1)^2$$
$$= k(k+1)(2k+1)/6 + (k+1)^2$$
$$= \frac{k(k+1)(2k+1)+6(k+1)^2}{6}$$
$$= \frac{(k+1)(k(2k+1)+6(k+1))}{6}$$
$$= \frac{(k+1)(2k^2+7k+6)}{6}$$
$$= \frac{(k+1)(2k+3)(k+2)}{6}$$
$$= \frac{(k+1)((k+1)+1))((2k+2)+1)}{6}$$

Ex. 3 Prove that we have

$1+3+5+7+\dots+(2n-1)=n^2$ for all positive integers n.

Proof:

Base step: For $n=1$ we have

$$2\cdot1-1=1^2$$
$$1=1$$

Induction step: For $n=k$, we assume

$$1+3+\dots+(2k-1)=k^2$$

For $n=k+1$, we have

$$1+3+\dots+(2(k+1)-1) = 1+\dots+(2k+1)$$
$$= k^2+(2k+1)$$
$$=(k+1)^2$$

Induction:

1. Prove for $n=1$
2. Assume the statement for $n=k$

3. Prove for $n=k+1$

$$K^2(4K^2-4K+1) + (2k+1)^3$$

$= 4K^4 - 4K^3 + K^2$

$= (2k+1)(4k^2+4k+1)$

$= 8k^3 + 8k^2 + 2k$

$+ \quad 4k^2 + 4k + 1$

$= 8k^3 + 12k^2 + 6k + 1$

$= 4k^4 + 4k^3 + 13k^2 + 6k + 1$

$= 2k^2 + K \quad + 1$