Nota | Order $\subset$ | or $\subseteq$ Other courses

strict | containment $\subsetneq$ | $\subset$

Set Containment Proofs

Example 1: Prove $(A\setminus B) \cup (C \setminus B) = (A \cup C \setminus B$

Let A, B be subsets of $\mathbb{R}^n$.

UNION: If $x \in A \cup B$, then $x \in A$ or $x \in B$

INTERSECTION: If $x \in A \cap B$, then $x \in A$ and $x \in B$.

If $x \in A$ implies $x \in B$, then $A \subset B$

A is contained in B

If $A \subset B$ and $B \subset A$, then $A = B$

$A \setminus B = \{x \in A : x \notin B\}$

Proof: First, we will prove $(A \setminus B) \cup (C \setminus B) \subset (A \cup C) \setminus B$.

Let $x \in (A \setminus B) \cup (C \setminus B)$. Then $x \in A \setminus B$ or $x \in C \setminus B$.

- If $x \in A \setminus B$, then $x \in A$ ~~and~~ $\cancel{x \notin B}$

(Fact:) Since $A \subset A \cup C$, we have $x \in A \cup C$ and $x \in B$.

- If $x \in C \setminus B$, then $x \in C$ and $x \in B$.

Since $C \subset A \cup C$, we have $x \in A \cup C$ and $x \notin B$.

In either case, we have $x \in A \cup C$ and $x \in B$.

So $x \in (A \cup C) \setminus B$.

Therefore, $(A \setminus B) \cup (C \setminus B) \subset (A \cup C) \setminus B$

Next, we will prove $(A \cup C) \setminus B \subset (A \setminus B) \cup (C \setminus B)$

Let $x \in (A \cup C) \setminus B$. Then $x \in A \cup C$ and $x \in B$.

So we have ~~$x \in A$ and $x \notin B$, or $x \in C$ and $x \notin B$.~~

$x \in A$ or $x \in C$, and $x \notin B$, and so

$x \in A$ and $x \in B$, or $x \in C$ and $x \notin B$.

So $x \in A \setminus B$, or $x \in C \setminus B$

Therefore, $x \in (A \setminus B) \cup (C \setminus B)$.

So $(A \cup C) \setminus B \subset (A \setminus B) \cup (C \setminus B)$.

$\Rightarrow$ Since the 2 sets are contained in each other, we conclude

$(A \setminus B) \cup (C \setminus B) = (A \cup C) \setminus B$.

Quiz 1 Review.

Universal quantifier — informal symbol

for all, for every, for each — $\forall$

Existential quantifier

There Exists, for some, for one — $\exists$

Examples

Definition of continuity

- For all $\varepsilon > 0$, there exists $\delta_\varepsilon > 0$ such that, for all $x, c \in \mathbb{R}$ with c fixed, if $|x - c| < \delta$, then $|f(x) - f(c)| < \varepsilon$
- For all students in Math 131 this summer, there exists an instructor running the course.
- For all TV programs on NBC, there exists the Ellen ~~Degeneres~~ Degeneres show.

Odd and Even Integers

Let a and b be even integers

Prove that $a+b$, $a-b$, ab are also even integers.

Odd: An integer n is odd if there exists some integer k such that $n = 2k+1$

Even: An integer n is even if there exists some integer k such that $n = 2k$.

proof: Since a and b are even integers, there exists integers k and ℓ such that $a=2k$ and $b=2\ell$. So we have

$$a+b = 2k+2\ell = 2(k+\ell),$$
$$a-b = 2k-2\ell = 2(k-\ell),$$
$$ab = (2k)(2\ell) = 4k\ell = 2(2k\ell).$$

Since $k+\ell$, $k-\ell$, and $2k\ell$ are also integers, we conclude that $a+b$, $a-b$, and ab is even.

Use a counterexample to show that $\frac{a}{b}$ with $b \neq 0$ does not have to be an even integer.

Counterexample: Let $a=6$ and $b=2$. Then a and b are even.

But $\frac{a}{b} = \frac{6}{2} = 3$, which is odd.

So $\frac{a}{b}$ is not necessarily an even integer.

Let a and b be even integers. Prove that $a+b+3$ is an odd integer.

We have

$$\begin{aligned} a+b+3 &= 2k+2\ell+3 \\ &= 2k+2\ell+2+1 \\ &= 2(k+\ell+1)+1 \end{aligned}$$

Since $k+\ell+1$ is also an integer, we conclude that $a+b+3$ is odd

Let $f: \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = x^2 - 4x + 4$$

Prove that $f(x) \geq 0$ for all $x \in \mathbb{R}$.

proof: . For all $x \in \mathbb{R}$, we have

$$f(x) = x^2 - 4x + 4 = (x-2)^2$$

which is ≥ 0

Prove that $f(x) > 0$ for all $x \in \mathbb{R}$.

proof: For all $x \in \mathbb{R}$ we have

$$f(x) = x^2 - 2x + 2$$

complete the square $= x^2 - 2x + 1 + 1$

$$= (x-1)^2 + 1$$

$$\geq 0 + 1$$

$$= 1$$

$$> 0.$$