MATH 131: Linear Algebra I

University of California, Riverside

Final Examination

Time limit: 3 hours

Score: _____ / 200

July 27, 2019

No resources can be used for the final examination, and talking to classmates is prohibited. You must take this examination to avoid receiving an automatic failing grade in the course.

By writing my name and student ID number below, I agree to the following terms:

- I promise not to engage in any form of academic dishonesty. In particular, I will not use any resources whatsoever. I understand that any act of cheating may cause me to receive a failing grade in the course and further disciplinary action from the university.
- I will turn my cellular phone off and place it on the desk in front of me. If I do not have a cellular phone, I will notify the instructor before the start of the final examination.
- If I need to use the restroom during the final examination, then I must ask the instructor for permission. I cannot use the restroom for more than 10 minutes, more than once, or while another student is using the restroom. Also, I cannot take anything with me to the restroom. If I violate any of these policies, I understand that the instructor may dismiss me early and will only be graded for the work done.
- I will not open this booklet until the instructor tells the class to do so.

Name:____________________________ Student ID:____________________________

(25pts) 1. Match the key terms with their correct definitions. Write the letter in upper-case corresponding to the definition in the blank next to the key term. Each blank is worth 1 point, for a total of 26 points, which means you can receive 1 point of extra credit if all your answers to this question are correct.

______addition	A. There exist $a_1, \dots, a_m \in \mathbb{F}$, not all 0, such that $a_1 v_1 + \cdots + a_m v_m = 0$.
______basis	B. The linear map $\pi : V \to V/U$ defined by $\pi(v) = v + U$ for all $v \in V$.
______dimension	C. A vector space V' of all linear functionals on V; in other words, $V' = \mathcal{L}(V, \mathbb{F})$.
______dual basis	D. A set V along with an addition on V and a scalar multiplication on V such that the following properties hold: commutativity, associativity, additive identity, additive inverse, multiplicative identity, and distributive properties.
______dual map	E. A linear map $T : V \to \mathbb{F}$, or, equivalently, an element of $\mathcal{L}(V, \mathbb{F})$.
______dual space	F. There exist $a_0, \dots, a_m \in \mathbb{F}$ such that $p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_m z^m$ for all $z \in \mathbb{F}$.
______finite-dimensional	G. A rectangular array of elements of $\mathbb{F}$ with m rows and n columns.
______injective	H. If $T : V \to W$ is a map, then this is a subset of V consisting of all vectors $v \in V$ such that $Tv = 0$.
______invertible	I. A subset U of V that is also a vector space, using the same addition and scalar multiplication as on V.
______isomorphism	J. The set of all linear combinations of a list of vectors $v_1, \dots, v_m$ in V.
______linear combination	K. Some list of vectors in this type of vector space spans the space.
______linear functional	L. If $T \in \mathcal{L}(V, W)$, then $T' \in \mathcal{L}(W', V')$ is the linear map defined by $T'(\varphi) = \varphi \circ T$ for $\varphi \in W'$.
______linearly dependent	M. A list $v_1, \dots, v_m$ of vectors in V that is of the form $a_1 v_1 + \cdots + a_m v_m$ for some $a_1, \dots, a_m \in \mathbb{F}$.
______linearly independent	N. A function $T : V \to W$ that satisfies $T(u + v) = Tu + Tv$ and $T(\lambda v) = \lambda(Tv)$ for all $\lambda \in \mathbb{F}$ and for all $u, v \in V$.
______linear map	O. A function $T : V \to W$ satisfies range $T = W$.
______matrix	P. A function that assigns an element $u + v \in V$ to each pair of elements $u, v \in V$.
______null space	Q. The only choice of $a_1, \dots, a_m \in \mathbb{F}$ that satisfies $a_1 v_1 + \cdots + a_m v_m = 0$ is $a_1 = \cdots = a_m = 0$.
______polynomial	R. A function that assigns an element $\lambda v \in V$ to each $\lambda \in \mathbb{F}$ and each $v \in V$.
______quotient map	S. If $v_1, \dots, v_n$ is a basis of V, then this is the list of $\varphi_1, \dots, \varphi_n$ of elements of V', where each φ_j is the linear functional on V such that $\varphi_j(v_k) = 1$ if $k = j$ and $\varphi_j(v_k) = 0$ if $k \neq j$.
______quotient space	T. An invertible linear map.
______range	U. If $T : V \to W$ is a map, then this is a subset of W consisting of all vectors of the form Tv for some $v \in V$.
______scalar multiplication	V. A map $T \in \mathcal{L}(V, W)$ is called this if there exists a map $S \in \mathcal{L}(W, V)$ such that we have $ST = I_V$ and $TS = I_W$, where I_V and I_W are the respective identity maps on V and W.
______span	W. The set $V/U = \{v + U : v \in V\}$ of all affine subsets of V parallel to U.
______subspace	X. A function $T : V \to W$ that satisfies this property: $Tu = Tv$ implies $u = v$.
______surjective	Y. Length of any basis of a finite-dimensional vector space.
______vector space	Z. Every $v \in V$ can be written uniquely in the form $v = a_1 v_1 + \cdots + a_n v_n$ for some $a_1, \dots, a_n \in \mathbb{F}$.

(25pts) 2. Let V be a vector space and suppose that U_1, U_2, U_3 are subspaces of V. Answer each of the following questions with either a proof or a counterexample.

(8pts) a. Is the operation of addition on the subspaces of V commutative? In other words, is it true that we have

$$U_1 + U_2 = U_2 + U_1?$$

(8pts) b. Is the operation of addition on the subspaces of V associative? In other words, is it true that we have

$$(U_1 + U_2) + U_3 = U_1 + (U_2 + U_3)?$$

(9pts) c. Does the left distributive property for the intersections and sums of the subspaces of V hold? In other words, is it true that we have

$$U_1 \cap (U_2 + U_3) = U_1 \cap U_2 + U_1 \cap U_3?$$

(50pts) 3. (Double-weighted question) Let $n \geq 2$ be an integer. Define the following vectors in the vector space $\mathbb{R}^n$:

$$\begin{aligned}
v_1 &= (1, 1, 0, 0, 0, \ldots, 0, 0, 0),\\
v_2 &= (0, 1, 1, 0, 0, \ldots, 0, 0, 0),\\
v_3 &= (0, 0, 1, 1, 0, \ldots, 0, 0, 0),\\
v_4 &= (0, 0, 0, 1, 1, \ldots, 0, 0, 0),\\
&\vdots\\
v_{n-2} &= (0, 0, 0, 0, 0, \ldots, 1, 1, 0),\\
v_{n-1} &= (0, 0, 0, 0, 0, \ldots, 0, 1, 1).
\end{aligned}$$

Also consider for all $t \in \mathbb{R}$ the vector $w(t) \in \mathbb{R}^n$ defined by

$$w(t) = (t, 0, 0, 0, 0, \ldots, 0, 0, 1).$$

Note: To answer both parts of this question, you must apply directly the definitions of the terms "linearly independent" and "linearly dependent". While arranging the vectors in the list as a matrix and taking the determinant of that matrix constitutes a valid approach to the correct answers, zero credit will be given for that approach because the Axler textbook does not talk about determinants in the chapters we covered in our course.

(30pts) a. For which value(s) of $t \in \mathbb{R}$ is the list $v_1, \ldots, v_{n-1}, w(t)$ linearly independent (and therefore a basis of $\mathbb{R}^n$)? Prove your answer.

This is extra space for the rest of your work for Question 3(a), if needed.

(20pts) b. For which value(s) of $t \in \mathbb{R}$ is the list $v_1, \ldots, v_{n-1}, w(t)$ linearly dependent? Prove your answer.

(25pts) 4. Let V be a vector space, and suppose we have $T \in \mathcal{L}(V) = \mathcal{L}(V, V)$. A scalar $\lambda \in \mathbb{F}$ is called an *eigenvalue* of T if there exists a corresponding *eigenvector*, which is a vector $v \in V$ that satisfies $v \neq 0$ and

$$Tv = \lambda v.$$

Suppose also that, for all $a \in \mathbb{F}$, there exist vectors $u, w \in V$ that satisfy

$$Tu = aw$$

and

$$Tw = au.$$

Prove that a and $-a$ are both eigenvalues of T.

Remark: If $a = 0$, then T has of course only one eigenvalue instead of two eigenvalues.

Hint: Work with the vectors $u + w, u - w \in V$.

(25pts) 5. Let n be a nonnegative integer. Define the map $T : \mathcal{P}_n(\mathbb{R}) \to \mathbb{R}^{n+1}$ by

$$Tp = \left(\int_0^1 p(x)\,dx, \int_2^3 p(x)\,dx, \int_4^5 p(x)\,dx, \ldots, \int_{2n-2}^{2n-1} p(x)\,dx, \int_{2n}^{2n+1} p(x)\,dx \right).$$

Show that T is an isomorphism.

Note: For your proof of this question, you may use without their proofs the following results:

- The sets $\operatorname{null} T$ and $\operatorname{range} T$ are subspaces of $\mathbb{F}^n$.
- (Fundamental Theorem of Linear Maps) If V is a finite-dimensional vector space and T is a linear map on V, then $\operatorname{range} T$ is finite-dimensional and $\dim V = \dim \operatorname{null} T + \dim \operatorname{range} T$.
- (Fundamental Theorem of Algebra) Every non-constant polynomial of degree n has at most n distinct roots on $\mathbb{R}$; that is, for each $p \in \mathcal{P}_n(\mathbb{R})$ with $p \neq 0$, there exist at most n distinct values of $x \in \mathbb{R}$ that satisfy $p(x) = 0$.
- A linear map T is invertible if and only if it is injective and surjective.
- A linear map T is injective if and only if $\operatorname{null} T = \{0\}$.
- If U is a subspace of $\mathbb{F}^n$ that satisfies $\dim U = \dim \mathbb{F}^n$, then $U = \mathbb{F}^n$.

(50pts) 6. (Double-weighted question) Suppose V is a finite-dimensional vector space and U is a subspace of V. Recall that

$$U^0 = \{\varphi \in V' : \varphi(u) = 0 \text{ for all } u \in U\}$$

is the annihilator of U. We will prove in two different ways that the dimension of U^0 is the difference of the dimension of V and the dimension of U.

(25pts) a. Use the inclusion map $i \in \mathcal{L}(U, V)$—defined by $i(u) = u$ for all $u \in U$—to establish

$$\dim U + \dim U^0 = \dim V.$$

Note: For your proof of this part of the question, you may use without their proofs the following results:

- (Fundamental Theorem of Linear Maps) If V is a finite-dimensional vector space and T is a linear map on V, then range T is finite-dimensional and $\dim V = \dim \operatorname{null} T + \dim \operatorname{range} T$.
- If V' is the dual space of V, then we have $\dim V' = \dim V$.
- If V is finite-dimensional, then every linear map on a subspace U of V can be extended to a linear map on V.
- If U is a subspace of V, then the dual space U' is also a subspace of V.
- The set range i' is a subspace of U'.

Hint: Apply the Fundamental Theorem of Linear Maps to the dual map $i' \in \mathcal{L}(V', U')$ of the inclusion map i.

(25pts) b. Suppose m and n are positive integers that satisfy $m \leq n$. Choose $u_1, \ldots, u_m$ to be a basis of U, and extend it to a basis $u_1, \ldots, u_m, \ldots, u_n$ of V. Let $\varphi_1, \ldots, \varphi_m, \ldots, \varphi_n$ be the corresponding dual basis of V'. Prove that $\varphi_{m+1}, \ldots, \varphi_n$ is a basis of U^0. Then find the dimensions of U, U^0, V to conclude

$$\dim U + \dim U^0 = \dim V.$$