MATH 131: Linear Algebra I
University of California, Riverside
Final Examination Solutions
July 27, 2019

(25pts) 1. Match the key terms with their correct definitions. Write the letter in upper-case corresponding to the definition in the blank next to the key term. Each blank is worth 1 point, for a total of 26 points, which means you can receive 1 point of extra credit if all your answers to this question are correct.

P addition	A. There exist $a_1, \ldots, a_m \in \mathbb{F}$, not all 0, such that $a_1v_1 + \cdots + a_mv_m = 0$.
Z basis	B. The linear map $\pi : V \to V/U$ defined by $\pi(v) = v + U$ for all $v \in V$.
Y dimension	C. A vector space V' of all linear functionals on V; in other words, $V' = \mathcal{L}(V, \mathbb{F})$.
S dual basis	D. A set V along with an addition on V and a scalar multiplication on V such that the following properties hold: commutativity, associativity, additive identity, additive inverse, multiplicative identity, and distributive properties.
L dual map	E. A linear map $T : V \to \mathbb{F}$, or, equivalently, an element of $\mathcal{L}(V, \mathbb{F})$.
C dual space	F. There exist $a_0, \ldots, a_m \in \mathbb{F}$ such that $p(z) = a_0 + a_1z + a_2z^2 + \cdots + a_mz^m$ for all $z \in \mathbb{F}$.
K finite-dimensional	G. A rectangular array of elements of $\mathbb{F}$ with m rows and n columns.
X injective	H. If $T : V \to W$ is a map, then this is a subset of V consisting of all vectors $v \in V$ such that $Tv = 0$.
V invertible	I. A subset U of V that is also a vector space, using the same addition and scalar multiplication as on V.
T isomorphism	J. The set of all linear combinations of a list of vectors $v_1, \ldots, v_m$ in V.
M linear combination	K. Some list of vectors in this type of vector space spans the space.
E linear functional	M. A list $v_1, \ldots, v_m$ of vectors in V that is of the form $a_1v_1 + \cdots + a_mv_m$ for some $a_1, \ldots, a_m \in \mathbb{F}$.
A linearly dependent	L. If $T \in \mathcal{L}(V, W)$, then $T' \in \mathcal{L}(W', V')$ is the linear map defined by $T'(\varphi) = \varphi \circ T$ for $\varphi \in W'$.
Q linearly independent	N. A function $T : V \to W$ that satisfies $T(u + v) = Tu + Tv$ and $T(\lambda v) = \lambda(Tv)$ for all $\lambda \in \mathbb{F}$ and for all $u, v \in V$.
N linear map	O. A function $T : V \to W$ satisfies $\operatorname{range} T = W$.
G matrix	P. A function that assigns an element $u + v \in V$ to each pair of elements $u, v \in V$.
H null space	Q. The only choice of $a_1, \ldots, a_m \in \mathbb{F}$ that satisfies $a_1v_1 + \cdots + a_mv_m = 0$ is $a_1 = \cdots = a_m = 0$.
F polynomial	R. A function that assigns an element $\lambda v \in V$ to each $\lambda \in \mathbb{F}$ and each $v \in V$.
B quotient map	S. If $v_1, \ldots, v_n$ is a basis of V, then this is the list of $\varphi_1, \ldots, \varphi_n$ of elements of V', where each φ_j is the linear functional on V such that $\varphi_j(v_k) = 1$ if $k = j$ and $\varphi_j(v_k) = 0$ if $k \neq j$.
W quotient space	T. An invertible linear map.
U range	U. If $T : V \to W$ is a map, then this is a subset of W consisting of all vectors of the form Tv for some $v \in V$.
R scalar multiplication	V. A map $T \in \mathcal{L}(V, W)$ is called this if there exists a map $S \in \mathcal{L}(W, V)$ such that we have $ST = I_V$ and $TS = I_W$, where I_V and I_W are the respective identity maps on V and W.
J span	W. The set $V/U = \{v + U : v \in V\}$ of all affine subsets of V parallel to U.
I subspace	X. A function $T : V \to W$ that satisfies this property: $Tu = Tv$ implies $u = v$.
O surjective	Y. Length of any basis of a finite-dimensional vector space.
D vector space	Z. Every $v \in V$ can be written uniquely in the form $v = a_1v_1 + \cdots + a_nv_n$ for some $a_1, \ldots, a_n \in \mathbb{F}$.

(25pts) 2. Let V be a vector space and suppose that U_1, U_2, U_3 are subspaces of V. Answer each of the following questions with either a proof or a counterexample.

(8pts) a. Is the operation of addition on the subspaces of V commutative? In other words, is it true that we have

$$U_1 + U_2 = U_2 + U_1?$$

Proof. This statement is true. Suppose we have $u \in U_1 + U_2$. Then we can write $u = u_1 + u_2$ for some $u_1 \in U_1$ and for some $u_2 \in U_2$. Since U_1 and U_2 are subspaces of V, we have in fact $u_1, u_2 \in V$. As V is a vector space, it satisfies in particular commutativity. So we have

$$\begin{aligned} u &= u_1 + u_2 \\ &= u_2 + u_1 \\ &\in U_2 + U_1, \end{aligned}$$

and so we get $U_1 + U_2 \subset U_2 + U_1$. To prove the other set containment, suppose we have $w \in U_2 + U_1$. Then we can write $w = w_2 + w_1$ for some $w_2 \in U_2$ and for some $w_1 \in U_1$. Since U_1 and U_2 are subspaces of V, we have in fact $w_1, w_2 \in V$. As V is a vector space, it satisfies in particular commutativity. So we have

$$\begin{aligned} w &= w_2 + w_1 \\ &= w_1 + w_2 \\ &\in U_1 + U_2, \end{aligned}$$

and so we get $U_2 + U_1 \subset U_1 + U_2$. So we conclude the set equality $U_1 + U_2 = U_2 + U_1$. □

(8pts) a. Is the operation of addition on the subspaces of V associative? In other words, is it true that we have

$$(U_1 + U_2) + U_3 = U_1 + (U_2 + U_3)?$$

Proof. This statement is true. Suppose we have $u \in (U_1 + U_2) + U_3$. Then we can write $u = (u_1 + u_2) + u_3$ for some $u_1 \in U_1$, for some $u_2 \in U_2$, and for some $u_3 \in U_3$. Since U_1, U_2, U_3 are subspaces of V, we have in fact $u_1, u_2, u_3 \in V$. As V is a vector space, it satisfies in particular associativity. So we have

$$\begin{aligned} u &= (u_1 + u_2) + u_3 \\ &= u_1 + (u_2 + u_3) \\ &\in U_1 + (U_2 + U_3), \end{aligned}$$

and so we get $(U_1 + U_2) + U_3 \subset U_1 + (U_2 + U_3)$. To prove the other set containment, suppose we have $w \in U_1 + (U_2 + U_3)$. Then we can write $w = w_1 + (w_2 + w_3)$ for some $w_1 \in U_1$, for some $w_2 \in U_2$, and for some $w_3 \in U_3$. Since U_1, U_2, U_3 are subspaces of V, we have in fact $w_1, w_2, w_3 \in V$. As V is a vector space, it satisfies in particular associativity. So we have

$$\begin{aligned} w &= w_1 + (w_2 + w_3) \\ &= (w_1 + w_2) + w_3 \\ &\in (U_1 + U_2) + U_3, \end{aligned}$$

and so we get $U_1 + (U_2 + U_3) \subset (U_1 + U_2) + U_3$. So we conclude the set equality $(U_1 + U_2) + U_3 = U_1 + (U_2 + U_3)$. □

(9pts) a. Does the left distributive property for the intersections and sums of the subspaces of V hold? In other words, is it true that we have

$$U_1 \cap (U_2 + U_3) = U_1 \cap U_2 + U_1 \cap U_3?$$

Proof. This statement is false. We will use a counterexample. Consider, for example, the vector space $V = \mathbb{R}^2$ with the usual operations of addition and scalar multiplication. And consider, for example, the following subsets of $\mathbb{R}^2$:

$$\begin{aligned} U_1 &= \{(x_1, x_2) \in \mathbb{R}^2 : x_1 = x_2\}, \\ U_2 &= \{(x_1, x_2) \in \mathbb{R}^2 : x_2 = 0\}, \\ U_3 &= \{(x_1, x_2) \in \mathbb{R}^2 : x_1 = 0\}. \end{aligned}$$

We will prove that U_1, U_2, U_3 are subspaces of $\mathbb{R}^2$.

- Additive identity: Since $0 \in \mathbb{R}$, for all $x_1, x_2 \in \mathbb{R}$, we have

$$\begin{aligned} (x_1, x_2) + (0, 0) = (x_1 + 0, x_2 + 0) = (x_1, x_2), \\ (x_1, 0) + (0, 0) = (x_1 + 0, 0 + 0) = (x_1, 0), \\ (0, x_2) + (0, 0) = (0 + 0, x_2 + 0) = (0, x_2). \end{aligned}$$

  So we conclude $(0, 0) \in U_1$, $(0, 0) \in U_2$, and $(0, 0) \in U_3$, respectively, which means that U_1, U_2, U_3 all contain the additive identity.

- Closed under addition: For all $x_1, x_2, y_1, y_2 \in \mathbb{R}$, we have

$$\begin{aligned}(x_1, x_2) + (y_1 + y_2) &= (x_1 + y_1, x_2 + y_2) \in U_1,\\ (x_1, 0) + (y_1, 0) &= (x_1 + y_1, 0 + 0) = (x_1 + y_1, 0) \in U_2,\\ (0, x_2) + (0, y_2) &= (0 + 0, x_2 + y_2) = (0, x_2 + y_2) \in U_3.\end{aligned}$$

  So U_1, U_2, U_3 are all closed under addition.
- Closed under scalar multiplication: For all $\lambda \in \mathbb{F}$ and for all $x_1, x_2 \in \mathbb{R}$, we have

$$\begin{aligned}\lambda(x_1, x_2) &= (\lambda x_1, \lambda x_2) \in U_1,\\ \lambda(x_1, 0) &= (\lambda x_1, \lambda \cdot 0) = (\lambda x_1, 0) \in U_2,\\ \lambda(0, x_2) &= (\lambda \cdot 0, \lambda x_2) = (0, \lambda x_2) \in U_3.\end{aligned}$$

  So U_1, U_2, U_3 are all closed under scalar multiplication.

Therefore, U_1, U_2, U_3 as defined above are all subspaces of $\mathbb{R}^2$. Now, let $u_1 \in U_1$ be arbitrary. Then we have $u_1 = (x_1, x_2)$ for some $x_1, x_2 \in \mathbb{R}$. In fact, we have

$$\begin{aligned}u_1 &= (x_1, x_2)\\ &= (x_1, 0) + (0, x_2)\\ &\in U_1 + U_2,\end{aligned}$$

and so we have $U_1 \subset U_2 + U_3$. By elementary set theory, this set containment implies the set equality

$$U_1 \cap (U_2 + U_3) = U_1.$$

On the other hand, if we consider the vectors $u \in U_1 \cap U_2 = \{(0, 0)\}$ and $v \in U_1 \cap U_3 = \{(0, 0)\}$, then we must have $u = (0, 0)$ and $v = (0, 0)$, and so we get

$$\begin{aligned}U_1 \cap U_2 + U_1 \cap U_3 &= \{u + v : u \in U_1 \cap U_2, v \in U_1 \cap U_3\}\\ &= \{(0, 0) + (0, 0) : (0, 0) \in U_1 \cap U_2, (0, 0) \in U_1 \cap U_3\}\\ &= \{(0 + 0, 0 + 0) : (0, 0) \in U_1 \cap U_2, (0, 0) \in U_1 \cap U_3\}\\ &= \{(0, 0) : (0, 0) \in U_1 \cap U_2, (0, 0) \in U_1 \cap U_3\}\\ &= \{(0, 0)\}.\end{aligned}$$

Therefore, as we have $U_1 \neq \{(0, 0)\}$, we conclude $U_1 \cap (U_2 + U_3) \neq U_1 \cap U_2 + U_1 \cap U_3$, signifying that the left distributive property does not necessarily hold. □

(50pts) 3. (Double-weighted question) Let $n \geq 2$ be an integer. Define the following vectors in the vector space $\mathbb{R}^n$:

$$\begin{aligned}v_1 &= (1, 1, 0, 0, 0, \ldots, 0, 0, 0),\\ v_2 &= (0, 1, 1, 0, 0, \ldots, 0, 0, 0),\\ v_3 &= (0, 0, 1, 1, 0, \ldots, 0, 0, 0),\\ v_4 &= (0, 0, 0, 1, 1, \ldots, 0, 0, 0),\\ &\vdots\\ v_{n-2} &= (0, 0, 0, 0, 0, \ldots, 1, 1, 0),\\ v_{n-1} &= (0, 0, 0, 0, 0, \ldots, 0, 1, 1).\end{aligned}$$

Also consider for all $t \in \mathbb{R}$ the vector $w(t) \in \mathbb{R}^n$ defined by

$$w(t) = (t, 0, 0, 0, 0, \ldots, 0, 0, 1).$$

Note: To answer both parts of this question, you must apply directly the definitions of the terms "linearly independent" and "linearly dependent". While arranging the vectors in the list as a matrix and taking the determinant of that matrix constitutes a valid approach to the correct answers, zero credit will be given for that approach because the Axler textbook does not talk about determinants in the chapters we covered in our course.

(30pts) a. For which value(s) of $t \in \mathbb{R}$ is the list $v_1, \ldots, v_{n-1}, w(t)$ linearly independent (and therefore a basis of $\mathbb{R}^n$)? Prove your answer.

Proof. Suppose there exist $a_1, \ldots, a_{n-1}, b \in \mathbb{F}$ that satisfy

$$a_1 v_1 + \cdots + a_{n-1} v_{n-1} + b w(t) = (0, \ldots, 0).$$

On the other hand, we have

$$\begin{aligned}
a_1v_1 + \cdots + a_{n-1}v_{n-1} + bw(t) &= a_1(1, 1, 0, 0, 0, \ldots, 0, 0, 0) + a_2(0, 1, 1, 0, 0, \ldots, 0, 0, 0) + a_3(0, 0, 1, 1, 0, \ldots, 0, 0, 0)\\
&\quad + a_4(0, 0, 0, 1, 1, \ldots, 0, 0, 0) + \cdots + a_{n-2}(0, 0, 0, 0, 0, \ldots, 1, 1, 0)\\
&\quad + a_{n-1}(0, 0, 0, 0, 0, \ldots, 0, 1, 1) + b(t, 0, 0, 0, 0, \ldots, 0, 0, 1)\\
&= (a_1, a_1, 0, 0, 0, \ldots, 0, 0, 0) + (0, a_2, a_2, 0, 0, \ldots, 0, 0, 0) + (0, 0, a_3, a_3, 0, \ldots, 0, 0, 0)\\
&\quad + (0, 0, 0, a_4, a_4, \ldots, 0, 0, 0) + \cdots + (0, 0, 0, 0, 0, \ldots, a_{n-2}, a_{n-2}, 0)\\
&\quad + (0, 0, 0, 0, 0, \ldots, 0, a_{n-1}, a_{n-1}) + (bt, 0, 0, 0, 0, \ldots, 0, 0, b)\\
&= (a_1 + bt, a_1 + a_2, a_2 + a_3, a_3 + a_4, \ldots, a_{n-3} + a_{n-2}, a_{n-2} + a_{n-1}, a_{n-1} + b).
\end{aligned}$$

Therefore, we obtain the equality

$$a_1v_1 + \cdots + a_{n-1}v_{n-1} + bw(t) = (0, 0, 0, 0, 0, \ldots, 0, 0, 0),$$

from which we can equate the coordinates of both sides to obtain the following system of equations

$$\begin{aligned}
a_1 + bt &= 0,\\
a_1 + a_2 &= 0,\\
a_2 + a_3 &= 0,\\
a_3 + a_4 &= 0,\\
&\vdots\\
a_{n-3} + a_{n-2} &= 0,\\
a_{n-2} + a_{n-1} &= 0,\\
a_{n-1} + b &= 0.
\end{aligned}$$

The first equation $a_1 + bt = 0$ implies $a_1 = -bt$. The second equation $a_1 + a_2 = 0$ with $a_1 = -bt$ implies $a_2 = bt$. The third equation $a_2 + a_3 = 0$ with $a_2 = bt$ implies $a_3 = -bt$. The fourth equation $a_3 + a_4 = 0$ with $a_3 = bt$ implies $a_4 = bt$. After continuing this process, we notice the pattern here: for each $j = 1, \ldots, n-2$, we have

$$a_j = \begin{cases} -bt & \text{if } j \text{ is odd,}\\ bt & \text{if } j \text{ is even.}\end{cases}$$

Writing all the n equations, we have

$$\begin{aligned}
a_1 &= -bt,\\
a_2 &= bt,\\
a_3 &= -bt,\\
a_4 &= bt,\\
&\vdots\\
a_{n-3} &= (-1)^{n-1}bt,\\
a_{n-2} &= (-1)^n bt,\\
a_{n-1} &= -b.
\end{aligned}$$

The equation $a_{n-2} = (-1)^n bt$ with one of our earlier equations $a_{n-2} + a_{n-1} = 0$ implies the resulting equation

$$\begin{aligned}
b((-1)^n t - 1) &= (-1)^n bt - b\\
&= (-1)^n bt + (-b)\\
&= a_{n-2} + a_{n-1}\\
&= 0.
\end{aligned}$$

For the list $v_1, \ldots, v_{n-1}, w(t)$ is linearly independent, we require that all scalars $a_1, \ldots, a_{n-1}, b$ be zero. In particular, we require $b = 0$. If $t \neq (-1)^n$, then the above equation $b((-1)^n t - 1) = 0$ implies $b = 0$, which in turn implies that all the scalars are zero:

$$a_1 = 0, \ldots, a_{n-1} = 0, b = 0.$$

Therefore, if $t \neq (-1)^n$, then $v_1, \ldots, v_{n-1}, w(t)$ is linearly independent. □

(20pts) b. For which value(s) of $t \in \mathbb{R}$ is the list $v_1, \ldots, v_{n-1}, w(t)$ linearly dependent? Prove your answer.

Proof. We found in part (a) of this question that, if $t \neq (-1)^n$, then the list $v_1, \ldots, v_{n-1}, w(t)$ is linearly independent. On the other hand, we claim that, if $t = (-1)^n$, then the list $v_1, \ldots, v_{n-1}, w(t)$ is linearly dependent. Suppose indeed that we have $t = (-1)^n$. To establish that the list is linearly dependent, we need to find $a_1, \ldots, a_{n-1}, b \in \mathbb{F}$, not all zero, that satisfy

$$a_1 v_1 + \cdots + a_{n-1} v_{n-1} + b w((-1)^n) = 0.$$

Choose, for example, $b = (-1)^n$, a value that coincides with $t = (-1)^n$. Then the solutions $a_1, \ldots, a_{n-1}, b$ to our system of equations in our proof of part (a) with $t = (-1)^n$ become

$$\begin{aligned}
a_1 &= -bt = -(-1)^n(-1)^n = -(-1)^{2n} = -1, \\
a_2 &= bt = (-1)^n(-1)^n = (-1)^{2n} = 1, \\
a_3 &= -bt = -(-1)^n(-1)^n = -(-1)^{2n} = -1, \\
a_4 &= bt = (-1)^n(-1)^n = (-1)^{2n} = 1, \\
&\vdots \\
a_{n-3} &= (-1)^{n-1} bt = (-1)^{n-1}(-1)^n(-1)^n = (-1)^{n-1}(-1)^{2n} = (-1)^{n-1}, \\
a_{n-2} &= (-1)^n bt = (-1)^n(-1)^n(-1)^n = (-1)^n(-1)^{2n} = (-1)^n, \\
a_{n-1} &= -b = -(-1)^n = (-1)^{n+1} = (-1)^{n-1}, \\
b &= (-1)^n.
\end{aligned}$$

In particular, not only is our above choice of $b \in \mathbb{F}$ nonzero, but also all the elements $a_1, \ldots, a_n, b \in \mathbb{F}$, are all nonzero. And they satisfy

$$\begin{aligned}
a_1 + b(-1)^n &= (-1) + (-1)^n(-1)^n = 0, \\
a_1 + a_2 &= (-1) + 1 = 0, \\
a_2 + a_3 &= 1 + (-1) = 0, \\
a_3 + a_4 &= (-1) + 1 = 0, \\
&\vdots \\
a_{n-3} + a_{n-2} &= (-1)^{n-1} + (-1)^n = 0, \\
a_{n-2} + a_{n-1} &= (-1)^n + (-1)^{n-1} = 0, \\
a_{n-1} + b &= (-1)^{n-1} + (-1)^n = 0.
\end{aligned}$$

So we have

$$\begin{aligned}
a_1 v_1 + \cdots + a_{n-1} v_{n-1} + b w((-1)^n) &= a_1(1,1,0,0,0,\ldots,0,0,0) + a_2(0,1,1,0,0,\ldots,0,0,0) + a_3(0,0,1,1,0,\ldots,0,0,0) \\
&\quad + a_4(0,0,0,1,1,\ldots,0,0,0) + \cdots + a_{n-2}(0,0,0,0,0,\ldots,1,1,0) \\
&\quad + a_{n-1}(0,0,0,0,0,\ldots,0,1,1) + b((-1)^n,0,0,0,0,\ldots,0,0,1) \\
&= (a_1,a_1,0,0,0,\ldots,0,0,0) + (0,a_2,a_2,0,0,\ldots,0,0,0) + (0,0,a_3,a_3,0,\ldots,0,0,0) \\
&\quad + (0,0,0,a_4,a_4,\ldots,0,0,0) + \cdots + (0,0,0,0,0,\ldots,a_{n-2},a_{n-2},0) \\
&\quad + (0,0,0,0,0,\ldots,0,a_{n-1},a_{n-1}) + (b(-1)^n,0,0,0,0,\ldots,0,0,b) \\
&= (a_1 + b(-1)^n, a_1 + a_2, a_2 + a_3, a_3 + a_4, \ldots, a_{n-3} + a_{n-2}, a_{n-2} + a_{n-1}, a_{n-1} + b) \\
&= (0,0,0,0,0,\ldots,0,0,0).
\end{aligned}$$

Therefore, if $t = (-1)^n$, then the list $v_1, \ldots, v_{n-1}, w(t)$ is linearly dependent. □

(25pts) 4. Let V be a vector space, and suppose we have $T \in \mathcal{L}(V) = \mathcal{L}(V,V)$. A scalar $\lambda \in \mathbb{F}$ is called an *eigenvalue* of T if there exists a corresponding *eigenvector*, which is a vector $v \in V$ that satisfies $v \neq 0$ and

$$Tv = \lambda v.$$

Suppose also that, for all $a \in \mathbb{F}$, there exist vectors $u, w \in V$ that satisfy

$$Tu = aw$$

and

$$Tw = au.$$

Prove that a and $-a$ are both eigenvalues of T.

Remark: If $a = 0$, then T has of course only one eigenvalue instead of two eigenvalues.

Hint: Work with the vectors $u + w, u - w \in V$.

Proof. For all $u, w \in V$, we have

$$\begin{aligned} T(v+w) &= Tv + Tw \\ &= aw + av \\ &= a(w+v) \\ &= a(v+w) \end{aligned}$$

and

$$\begin{aligned} T(v-w) &= Tv - Tw \\ &= aw - av \\ &= a(w-v) \\ &= -a(v-w). \end{aligned}$$

Next, we claim that, if $v \neq 0$ and $w \neq 0$, then $v + w \neq 0$ and $v - w \neq 0$. It is much easier to prove the contrapositive of our claim: If $v + w = 0$ or $v - w = 0$, then $v = 0$ and $w = 0$. If $v + w = 0$, then $v = -w$, which means

$$\begin{aligned} -2w &= -w - w \\ &= v - w \\ &= 0, \end{aligned}$$

from which we get $w = 0$, and in turn $v = -w = 0$. Similarly, if $v - w = 0$, then $v = w$, which means

$$\begin{aligned} 2w &= w + w \\ &= v + w \\ &= 0, \end{aligned}$$

from which we get $w = 0$, and in turn $v = w = 0$. This completes the proof of the the contrapositive of our claim and hence our claim itself. Therefore, if $v + w$ and $v - w$ are nonzero, it follows that a and $-a$ are eigenvalues of T, with $v + w$ and $v - w$ being their respective eigenvectors. □

(25pts) 5. Let $n \geq 0$ be an integer. Define the map $T : \mathcal{P}_n(\mathbb{R}) \to \mathbb{R}^{n+1}$ by

$$Tp = \left(\int_0^1 p(x)\,dx, \int_2^3 p(x)\,dx, \int_4^5 p(x)\,dx, \ldots, \int_{2n-2}^{2n-1} p(x)\,dx, \int_{2n}^{2n+1} p(x)\,dx \right).$$

Show that T is an isomorphism.

Note: For your proof of this question, you may use without their proofs the following results:

- The sets $\operatorname{null} T$ and $\operatorname{range} T$ are subspaces of $\mathbb{F}^n$.
- (Fundamental Theorem of Linear Maps) If V is a finite-dimensional vector space and T is a linear map on V, then $\operatorname{range} T$ is finite-dimensional and $\dim V = \dim \operatorname{null} T + \dim \operatorname{range} T$.
- (Fundamental Theorem of Algebra) Every non-constant polynomial of degree n has at most n distinct roots on $\mathbb{R}$; that is, for each $p \in \mathcal{P}_n(\mathbb{R})$ with $p \neq 0$, there exist at most n distinct values of $x \in \mathbb{R}$ that satisfy $p(x) = 0$.
- A linear map T is invertible if and only if it is injective and surjective.
- A linear map T is injective if and only if $\operatorname{null} T = \{0\}$.
- If U is a subspace of $\mathbb{F}^n$ that satisfies $\dim U = \dim \mathbb{F}^n$, then $U = \mathbb{F}^n$.

Proof. To show that $T : \mathcal{P}_n(\mathbb{R}) \to \mathbb{R}^{n+1}$ is an isomorphism, we need to show that T is linear and invertible. First, we will show that T is linear.

- Additivity: For all $p, q \in \mathcal{P}_n(\mathbb{R})$, we have

$$\begin{aligned}
T(p+q) &= \left(\int_0^1 (p+q)(x)\,dx, \int_2^3 (p+q)(x)\,dx, \int_4^5 (p+q)(x)\,dx, \dots, \int_{2n-2}^{2n-1} (p+q)(x)\,dx, \int_{2n}^{2n+1} (p+q)(x)\,dx\right) \\
&= \left(\int_0^1 p(x)+q(x)\,dx, \int_2^3 p(x)+q(x)\,dx, \int_4^5 p(x)+q(x)\,dx,\right. \\
&\qquad\qquad \left.\dots, \int_{2n-2}^{2n-1} p(x)+q(x)\,dx, \int_{2n}^{2n+1} p(x)+q(x)\,dx\right) \\
&= \left(\int_0^1 p(x)\,dx + \int_0^1 q(x)\,dx, \int_2^3 p(x)\,dx + \int_2^3 q(x)\,dx, \int_4^5 p(x)\,dx + \int_4^5 q(x)\,dx,\right. \\
&\qquad\qquad \left.\dots, \int_{2n-2}^{2n-1} p(x)\,dx + \int_{2n-2}^{2n-1} q(x)\,dx, \int_{2n}^{2n+1} p(x)\,dx + \int_{2n}^{2n+1} q(x)\,dx\right) \\
&= \left(\int_0^1 p(x)\,dx, \int_2^3 p(x)\,dx, \int_4^5 p(x)\,dx, \dots, \int_{2n-2}^{2n-1} p(x)\,dx, \int_{2n}^{2n+1} p(x)\,dx\right) \\
&\quad + \left(\int_0^1 q(x)\,dx, \int_2^3 q(x)\,dx, \int_4^5 q(x)\,dx, \dots, \int_{2n-2}^{2n-1} q(x)\,dx, \int_{2n}^{2n+1} q(x)\,dx\right) \\
&= Tp + Tq.
\end{aligned}$$

- Homogeneity: For all $\lambda \in \mathbb{F}$ and for all $p \in \mathcal{P}_n(\mathbb{R})$, we have

$$\begin{aligned}
T(\lambda p) &= \left(\int_0^1 (\lambda p)(x)\,dx, \int_2^3 (\lambda p)(x)\,dx, \int_4^5 (\lambda p)(x)\,dx, \dots, \int_{2n-2}^{2n-1} (\lambda p)(x)\,dx, \int_{2n}^{2n+1} (\lambda p)(x)\,dx\right) \\
&= \left(\int_0^1 \lambda p(x)\,dx, \int_2^3 \lambda p(x)\,dx, \int_4^5 \lambda p(x)\,dx, \dots, \int_{2n-2}^{2n-1} \lambda p(x)\,dx, \int_{2n}^{2n+1} \lambda p(x)\,dx\right) \\
&= \left(\lambda \int_0^1 p(x)\,dx, \lambda \int_2^3 p(x)\,dx, \lambda \int_4^5 p(x)\,dx, \dots, \lambda \int_{2n-2}^{2n-1} p(x)\,dx, \lambda \int_{2n}^{2n+1} p(x)\,dx\right) \\
&= \lambda \left(\int_0^1 p(x)\,dx, \int_2^3 p(x)\,dx, \int_4^5 p(x)\,dx, \dots, \int_{2n-2}^{2n-1} p(x)\,dx, \int_{2n}^{2n+1} p(x)\,dx\right) \\
&= \lambda Tp.
\end{aligned}$$

Since the two properties of a linear map are satisfied, this map is linear. Next, we will show that T is injective. Suppose we have $p \in \operatorname{null} T$, which means we have $Tp = (0, \dots, 0)$. Combining this with the original definition of T, we get

$$\left(\int_0^1 p(x)\,dx, \int_2^3 p(x)\,dx, \int_4^5 p(x)\,dx, \dots, \int_{2n-2}^{2n-1} p(x)\,dx, \int_{2n}^{2n+1} p(x)\,dx\right) = (0, 0, 0, \dots, 0, 0),$$

from which we can equate the coordinates of both sides to obtain

$$\begin{aligned}
\int_0^1 p(x)\,dx &= 0, \\
\int_2^3 p(x)\,dx &= 0, \\
\int_4^5 p(x)\,dx &= 0, \\
&\vdots \\
\int_{2n-2}^{2n-1} p(x)\,dx &= 0, \\
\int_{2n}^{2n+1} p(x)\,dx &= 0.
\end{aligned}$$

We claim that, if every one of these equations above holds, then we must have $p = 0$; that is, the polynomial p must be equal to the zero map. Suppose by contradiction that there exists a polynomial $p \in \mathcal{P}_n(\mathbb{R})$ with $p \neq 0$ that satisfies the above equations. Consider the interval $[2i, 2i+1] \subset \mathbb{R}$ for each $i = 0, 1, \dots, m$. Then there exists a nonempty subset $U_i \subsetneq [2i, 2i+1]$ with $U_i \neq [2i, 2i+1]$ such that p is positive on U_i and negative on its set complement $[2i, 2i+1] \setminus U_i$, in such a way that p also satisfies

$$\int_{2i}^{2i+1} p(x)\,dx = 0.$$

If p did not cross the x-axis on the interval $[2i, 2i+1]$, then p is either positive or negative on all of $[2i, 2i+1]$, meaning that we would have

$$\int_{2i}^{2i+1} p(x)\,dx \neq 0,$$

which contradicts the previous displayed equation. So p must cross the x-axis at least once on $[2i, 2i+1]$. In other words, there exists at least one root of p on $[2i, 2i+1]$ for all $i = 0, 1, \dots, n$. This means that there exist at least $n+1$ distinct roots of p on the union of intervals

$$[0,1] \cup [2,3] \cup [4,5] \cup \cdots \cup [2n-2, 2n-1] \cup [2n, 2n+1].$$

As such a union of intervals is a subset of $\mathbb{R}$, we can say more simply that there exist at least $n+1$ distinct roots of $p \in \mathcal{P}_n(\mathbb{R})$. However, this contradicts the Fundamental Theorem of Algebra, which states that every $p \in \mathcal{P}_n(\mathbb{R})$ with $p \neq 0$ has at most n distinct roots on $\mathbb{R}$. Therefore, we must have $p(x) = 0$ for all $x \in \mathbb{R}$ in order to satisfy the above equations without obtaining a contradiction. In other words, we must have $p = 0$, and so we get $\operatorname{null} T \subset \{0\}$. But we also have $T(0) = (0, \dots, 0)$, which means $\{0\} \subset \operatorname{null} T$. So we conclude the set equality $\operatorname{null} T = \{0\}$, and so T is injective. Next, we will prove that T is surjective. Since we have $\dim \mathcal{P}_n(\mathbb{R}) = n$, it follows that $\mathcal{P}_n(\mathbb{R})$ is finite-dimensional. So the Fundamental Theorem of Linear Maps asserts that $\operatorname{range} T$ is finite-dimensional and

$$\begin{aligned} \dim \operatorname{range} T &= \dim \mathcal{P}_n(\mathbb{R}) - \dim \operatorname{null} T \\ &= \dim \mathcal{P}_n(\mathbb{R}) - \dim\{0\} \\ &= (n+1) - 0 \\ &= n+1 \\ &= \dim \mathbb{R}^{n+1}. \end{aligned}$$

Since $\operatorname{range} T$ is a subspace of $\mathbb{R}^{n+1}$, we conclude $\operatorname{range} T = \mathbb{R}^{n+1}$, and so T is surjective. We have established that T is both injective and surjective, which means that T is invertible. Therefore, T is both linear and invertible, which means it is an isomorphism. □

(50pts) 6. (Double-weighted question) Suppose V is a finite-dimensional vector space and U is a subspace of V. Recall that

$$U^0 = \{\varphi \in V' : \varphi(u) = 0 \text{ for all } u \in U\}$$

is the annihilator of U. We will prove in two different ways that the dimension of U^0 is the difference of the dimension of V and the dimension of U.

(25pts) a. Use the inclusion map $i \in \mathcal{L}(U, V)$—defined by $i(u) = u$ for all $u \in U$—to establish

$$\dim U + \dim U^0 = \dim V.$$

Note: For your proof of this part of the question, you may use without their proofs the following results:

- (Fundamental Theorem of Linear Maps) If V is a finite-dimensional vector space and T is a linear map on V, then $\operatorname{range} T$ is finite-dimensional and $\dim V = \dim \operatorname{null} T + \dim \operatorname{range} T$.
- If V' is the dual space of V, then we have $\dim V' = \dim V$.
- If V is finite-dimensional, then every linear map on a subspace U of V can be extended to a linear map on V.
- If U is a subspace of V, then the dual space U' is also a subspace of V.
- The set $\operatorname{range} i'$ is a subspace of U'.

Hint: Apply the Fundamental Theorem of Linear Maps to the dual map $i' \in \mathcal{L}(V', U')$ of the inclusion map i.

Proof (3.106 of Axler). Suppose $i \in \mathcal{L}(U, V)$ is the inclusion map defined as given in the problem statement. First, we will prove $i' \in \mathcal{L}(V', U')$. Let $\varphi, \psi \in V'$ and $\lambda \in \mathbb{F}$ be arbitrary.

- Additivity: For all $u \in U$, we have

$$\begin{aligned} (i'(\varphi + \psi))(u) &= ((\varphi + \psi) \circ i)(u) \\ &= (\varphi + \psi)(i(u)) \\ &= (\varphi + \psi)(u) \\ &= \varphi(u) + \psi(u) \\ &= \varphi(i(u)) + \psi(i(u)) \\ &= (\varphi \circ i)(u) + (\psi \circ i)(u) \\ &= (\varphi \circ i + \psi \circ i)(u) \\ &= (i'(\varphi) + i'(\psi))(u). \end{aligned}$$

  So we conclude $i'(\varphi + \psi) = i'(\varphi) + i'(\psi)$.

- Homogeneity: For all $u \in U$, we have

$$\begin{aligned}(i'(\lambda\varphi))(u) &= ((\lambda\varphi) \circ i)(u) \\ &= (\lambda\varphi)(i(u)) \\ &= (\lambda\varphi)(u) \\ &= \lambda\varphi(u) \\ &= \lambda\varphi(i(u)) \\ &= \lambda(\varphi \circ i)(u) \\ &= (\lambda(\varphi \circ i))(u) \\ &= (\lambda i'(\varphi))(u).\end{aligned}$$

So we conclude $i'(\lambda\varphi) = \lambda i'(\varphi)$.

Since additivity and homogeneity of i' are satisfied, i' is linear. Next, we will prove $\operatorname{null} i' = U^0$. We have

$$\begin{aligned}\operatorname{null} i' &= \{\varphi \in V' : i'(\varphi) = 0\} \\ &= \{\varphi \in V' : \varphi \circ i = 0\} \\ &= \{\varphi \in V' : \varphi \circ i(u) = 0 \text{ for all } u \in U\} \\ &= \{\varphi \in V' : \varphi(u) = 0 \text{ for all } u \in U\} \\ &= U^0,\end{aligned}$$

as we claimed. Next, we will prove $\operatorname{range} i' = U$. Suppose we have $\varphi \in U'$. Since U is a subspace of V, we can extend φ to linear functional ψ on V. Now, for all $v \in V$, we have

$$\begin{aligned}(i'(\psi))(v) &= (\psi \circ i)(v) \\ &= \psi(i(v)) \\ &= \psi(v),\end{aligned}$$

from which we conclude $i'(\psi) = \psi$. In particular, for all $v \in U$, we have $\psi = \varphi$, and so we get $i'(\psi) = \psi = \varphi$, which means we have $\varphi \in \operatorname{range} i'$. Therefore, we obtain $U \subset \operatorname{range} i'$. But $\operatorname{range} i'$ is a subspace of U. So we conclude the set equality $\operatorname{range} i' = U$, as we claimed. Therefore, we have

$$\begin{aligned}\dim U + \dim U^0 &= \dim \operatorname{range} i' + \dim \operatorname{null} i' \\ &= \dim V' \\ &= \dim V,\end{aligned}$$

as desired. □

(25pts) b. Suppose m and n are positive integers that satisfy $m \leq n$. Choose $u_1, \ldots, u_m$ to be a basis of U, and extend it to a basis $u_1, \ldots, u_m, \ldots, u_n$ of V. Let $\varphi_1, \ldots, \varphi_m, \ldots, \varphi_n$ be the corresponding dual basis of V'. Prove that $\varphi_{m+1}, \ldots, \varphi_n$ is a basis of U^0. Then find the dimensions of U, U^0, V to conclude

$$\dim U + \dim U^0 = \dim V.$$

Proof. To prove that $\varphi_{m+1}, \ldots, \varphi_n$ is a basis of U^0, we need to prove that it is linearly independent and spans U^0. First, we will show that $\varphi_{m+1}, \ldots, \varphi_n$ is linearly independent. Suppose $c_{m+1}, \ldots, c_n \in \mathbb{F}$ satisfy

$$c_{m+1}\varphi_{m+1} + \cdots + c_n\varphi_n = 0.$$

Then we have

$$\begin{aligned}0 &= c_{m+1}\varphi_{m+1} + \cdots + c_n\varphi_n \\ &= 0\varphi_1 + \cdots + 0\varphi_m + c_{m+1}\varphi_{m+1} + \cdots + c_n\varphi_n.\end{aligned}$$

Since we assumed in the premises that $\varphi_1, \ldots, \varphi_m, \ldots, \varphi_n$ is the dual basis of V', it is linearly independent, which means all the scalars are zero. In particular, we have

$$c_{m+1} = 0, \ldots, c_n = 0.$$

Therefore, $\varphi_{m+1}, \ldots, \varphi_n$ is linearly independent. Next, we will show that $\varphi_{m+1}, \ldots, \varphi_n$ spans V. In other words, we need to prove $\operatorname{span}(\varphi_{m+1}, \ldots, \varphi_n) = V$. Suppose we have $\varphi \in \operatorname{span}(\varphi_{m+1}, \ldots, \varphi_m)$. Then we can write

$$\varphi = c_{m+1}\varphi_{m+1} + \cdots + c_n\varphi_n$$

for some $c_{m+1}, \ldots, c_n \in \mathbb{F}$. Additionally, since we assumed in the premises that $u_1, \ldots, u_m$ is a basis of U, we can write every $u \in U$ uniquely as

$$u = a_1u_1 + \cdots + a_mu_m$$

for some $a_1, \ldots, a_m \in \mathbb{F}$. So, for all $u \in U$, we have

$$\begin{aligned}
\varphi(u) &= \varphi(a_1u_1 + \cdots + a_mu_m) \\
&= (c_{m+1}\varphi_{m+1} + \cdots + c_n\varphi_n)(a_1u_1 + \cdots + a_mu_m) \\
&= c_{m+1}\varphi_{m+1}(a_1u_1 + \cdots + a_mu_m) + \cdots + c_n\varphi_n(a_1u_1 + \cdots + a_mu_m) \\
&= c_{m+1}(a_1\varphi_{m+1}(u_1) + \cdots + a_m\varphi_{m+1}(u_m)) + \cdots + c_n(a_1\varphi_n(u_1) + \cdots + a_m\varphi_n(u_m)) \\
&= c_{m+1}(a_1 \cdot 0 + \cdots + a_m \cdot 0) + \cdots + c_n(a_1 \cdot 0 + \cdots + a_m \cdot 0) \\
&= 0.
\end{aligned}$$

Therefore, we have $\varphi \in U^0$, and so we have $\operatorname{span}(\varphi_{m+1}, \ldots, \varphi_n) \subset U^0$. Now, to prove the other set containment, suppose that we have $\varphi \in U^0$. Since the dual basis $\varphi_1, \ldots, \varphi_m, \ldots, \varphi_n$ is a basis of the dual space V', we can write

$$\varphi = c_1\varphi_1 + \cdots + c_m\varphi_m + \cdots + c_n\varphi_n$$

for some $c_1, \ldots, c_m, \ldots, c_n \in \mathbb{F}$. For each $j = 1, \ldots, m$, we have $u_j \in U$ since $u_1, \ldots, u_m$ is a basis of U. So, for each $j = 1, \ldots, m$, we have

$$\begin{aligned}
0 &= \varphi(u_j) \\
&= (c_1\varphi_1 + \cdots + c_m\varphi_m + \cdots + c_n\varphi_n)(u_j) \\
&= (c_1\varphi_1 + \cdots + c_j\varphi_j \cdots + c_m\varphi_m + \cdots + c_n\varphi_n)(u_j) \\
&= c_1\varphi_1(u_j) + \cdots + c_j\varphi_j(u_j) + \cdots + c_m\varphi_m(u_j) + \cdots + c_n\varphi_n(u_j) \\
&= c_1 \cdot 0 + \cdots + c_j \cdot 1 + \cdots + c_m \cdot 0 + \cdots + c_n \cdot 0 \\
&= c_j.
\end{aligned}$$

In other words, we have

$$c_1 = 0, \ldots, c_m = 0,$$

and so we can write

$$\begin{aligned}
\varphi &= c_1\varphi_1 + \cdots + c_m\varphi_m + \cdots + c_n\varphi_n \\
&= c_1\varphi_1 + \cdots + c_m\varphi_m + c_{m+1}\varphi_{m+1} + \cdots + c_n\varphi_n \\
&= 0 \cdot \varphi_1 + \cdots + 0 \cdot \varphi_m + c_{m+1}\varphi_{m+1} + \cdots + c_n\varphi_n \\
&= c_{m+1}\varphi_{m+1} + \cdots + c_n\varphi_n.
\end{aligned}$$

Therefore, we have $\varphi \in \operatorname{span}(\varphi_{m+1}, \ldots, \varphi_n)$, and so we have $U^0 \subset \operatorname{span}(\varphi_{m+1}, \ldots, \varphi_n)$. So we conclude the desired set equality $\operatorname{span}(\varphi_{m+1}, \ldots, \varphi_n) = U^0$, which means $\varphi_{m+1}, \ldots, \varphi_n$ spans U^0. Therefore, we conclude that $\varphi_{m+1}, \ldots, \varphi_n$ is a basis of U^0. Furthermore, we get $\dim(U^0) = n - m$. Also, the assumptions of this problem imply $\dim U = m$ and $\dim V = n$. Therefore, we conclude

$$\begin{aligned}
\dim U + \dim U^0 &= m + (n - m) \\
&= n \\
&= \dim V,
\end{aligned}$$

as desired. □