MATH 131: Linear Algebra I
University of California, Riverside
Group Examination 1 Solutions
June 27, 2019

(20pts) 1. For this question, you will need to refer to the definitions in Chapter 1 of your Axler textbook to find the answers. (This is one of the reasons this examination is open book, open notes, open homework, and open classmates.)

(4pts) a. Write down the definitions of *addition* and *scalar multiplication* on a set V.

Definition. From Definition 1.18 of Axler:

- An *addition* on a set V is a function that assigns an element $u + v \in V$ to each pair of elements $u, v \in V$.
- A *scalar multiplication* on a set V is a function that assigns an element $\lambda v \in V$ to each $\lambda \in \mathbb{F}$ and each $v \in V$.

(12pts) b. Now assume that V is a *vector space* over a field $\mathbb{F}$. Write down all the properties of a vector space.

Definition. From Definition 1.19 of Axler: A *vector space* is a set V along with an addition on V and a scalar multiplication on V such that the following properties hold:

- *commutativity:* $u + v = v + u$ for all $u, v \in V$;
- *associativity:* $(u + v) + w = u + (v + w)$ and $(ab)v = a(bv)$ for all $u, v, w \in V$ and for all $a, b \in \mathbb{F}$;
- *additive identity:* there exists an element $0 \in V$ such that $v + 0 = v$ for all $v \in V$;
- *additive inverse:* for every $v \in V$, there exists $w \in V$ such that $v + w = 0$;
- *multiplicative identity:* $1v = v$ for all $v \in V$;
- *distributive properties:* $a(u + v) = au + av$ and $(a + b)v = av + bv$ for all $a, b \in \mathbb{F}$ and for all $u, v \in V$.

(4pts) c. Write down the definition of a *subspace* U of V.

Definition. From Definition 1.32 of Axler:

- A subset U of V is called a *subspace* of V if U is also a vector space, using the same addition and scalar multiplication as on V.

(20pts) 2. Let n be a positive integer, let $\lambda \in \mathbb{F}$ be a scalar, and let $x, y \in \mathbb{F}^n$ be lists of length n. Define on $\mathbb{F}^n$ the operations of "addition"

$$x \text{ ``+'' } y = x - y$$

and "scalar multiplication"

$$\lambda \text{ ``}\times\text{'' } x = -\lambda x.$$

Is $\mathbb{F}^n$ a vector space over $\mathbb{F}$ with respect to these operations? If so, prove it. If not, prove which of the properties of a vector space are satisfied and give counterexamples for the properties of a vector space that are not satisfied.

Proof. We will list all the properties, determine whether or not they are satisfied, and prove or give a counterexample to each property as appropriate. In the case of counterexamples, we will let $n = 2$, so that the list $(x_1, \ldots, x_n) \in \mathbb{F}^n$ becomes $(x_1, x_2) \in \mathbb{F}^2$; this is to ensure that our counterexamples are explicit.

- Commutativity is not satisfied. Let $(x_1, x_2) = (1, 1), (y_1, y_2) = (2, 2) \in \mathbb{F}^2$. Then

$$\begin{aligned}(x_1, x_2) \text{ ``+'' } (y_1, y_2) &= (x_1, x_2) - (y_1, y_2)\\ &= (1, 1) - (2, 2)\\ &= (-1, -1)\end{aligned}$$

and

$$\begin{aligned}(y_1, y_2) \text{ ``+'' } (x_1, x_2) &= (y_1, y_2) - (x_1, x_2)\\ &= (2, 2) - (1, 1)\\ &= (1, 1).\end{aligned}$$

Since we have $(-1, -1) \neq (1, 1)$, we conclude $(x_1, x_2) \text{ ``+'' } (y_1, y_2) \neq (y_1, y_2) \text{ ``+'' } (x_1, x_2)$.

- Associativity is not satisfied. Let $(x_1, x_2) = (1, 1), (y_1, y_2) = (2, 2), (z_1, z_2) = (3, 3) \in \mathbb{F}^2$. Then

$$\begin{aligned}((x_1, x_2) \text{ ``+'' } (y_1, y_2)) \text{ ``+'' } (z_1, z_2) &= ((x_1, x_n) - (y_1, y_2)) \text{ ``+'' } (z_1, z_2)\\ &= ((x_1, x_2) - (y_1, y_2)) - (z_1, z_2)\\ &= (x_1, x_2) - (y_1, y_2) - (z_1, z_2)\\ &= (1, 1) - (2, 2) - (3, 3)\\ &= (-4, -4)\end{aligned}$$

and

$$\begin{aligned}(x_1, x_2) \text{ “+” } ((y_1, y_2) \text{ “+” } (z_1, z_2)) &= (x_1, x_2) \text{ “+” } ((y_1, y_2) - (z_1, z_2)) \\ &= (x_1, x_2) - ((y_1, y_2) - (z_1, z_2)) \\ &= (x_1, x_2) - (y_1, y_2) + (z_1, z_2) \\ &= (1, 1) - (2, 2) + (3, 3) \\ &= (2, 2).\end{aligned}$$

Since we have $(-4, -4) \neq (2, 2)$, we conclude $((x_1, x_2) \text{ “+” } (y_1, y_2)) \text{ “+” } (z_1, z_2) \neq (x_1, x_2) \text{ “+” } ((y_1, y_2) \text{ “+” } (z_1, z_2))$.

- Additive identity is satisfied. Suppose we have $(0, 0) \in \mathbb{F}^2$. Then, for all $(x_1, x_2) \in \mathbb{F}^2$, we have

$$\begin{aligned}(x_1, x_2) \text{ “+” } (0, 0) &= (x_1, x_2) - (0, 0) \\ &= (x_1 - 0, x_2 - 0) \\ &= (x_1, x_2).\end{aligned}$$

- Additive inverse is satisfied. For all $(x_1, x_2) \in \mathbb{F}^2$, we have

$$\begin{aligned}(x_1, x_2) \text{ “+” } (x_1, x_2) &= (x_1, x_2) - (x_1, x_2) \\ &= (x_1 - x_1, x_2 - x_2) \\ &= (0, 0).\end{aligned}$$

- Multiplicative identity is not satisfied. Let $\lambda = 1 \in \mathbb{F}$ and $(x_1, x_2) = (1, 1) \in \mathbb{F}^2$. Then

$$\begin{aligned}\lambda \text{ “×” } (x_1, x_2) &= -\lambda(x_1, x_2) \\ &= -1(1, 1) \\ &= (-1, -1)\end{aligned}$$

and

$$\begin{aligned}\lambda(x_1, x_2) &= 1(1, 1) \\ &= (1, 1).\end{aligned}$$

Since we have $(-1, 1) \neq (1, 1)$, we conclude $\lambda \text{ “×” } (x_1, x_2) \neq \lambda(x_1, x_2)$.

- Distributive properties are not satisfied. It is true that the operations satisfy the left distributive property: for all $a, b \in \mathbb{F}$ and for all $(x_1, \dots, x_n), (y_1, \dots, y_n) \in \mathbb{F}^n$, we have

$$\begin{aligned}a \text{ “×” } ((x_1, \dots, x_n) \text{ “+” } (y_1, \dots, y_n)) &= a \text{ “×” } ((x_1, \dots, x_n) - (y_1, \dots, y_n)) \\ &= a \text{ “×” } (x_1 - y_1, \dots, x_n - y_n) \\ &= -a(x_1 - y_1, \dots, x_n - y_n) \\ &= -a((x_1, \dots, x_n) - (y_1, \dots, y_n)) \\ &= (-a(x_1, \dots, x_n)) - (-a(y_1, \dots, y_n)) \\ &= (a \text{ “×” } (x_1, \dots x_n)) - (a \text{ “×” } (y_1, \dots, y_n)) \\ &= (a \text{ “×” } (x_1, \dots x_n)) \text{ “+” } (a \text{ “×” } (y_1, \dots, y_n)).\end{aligned}$$

However, the operations do not satisfy the right distributive property. Let $a = 1, b = 2 \in \mathbb{F}$ and $(x_1, \dots, x_n) = (1, \dots, 1) \in \mathbb{F}^n$. Then we have

$$\begin{aligned}(a + b) \text{ “×” } (x_1, x_2) &= -(a + b)(x_1, x_2) \\ &= -(1 + 2)(1, 1) \\ &= (-3, -3)\end{aligned}$$

and

$$\begin{aligned}(a \text{ “×” } (x_1, x_2)) \text{ “+” } (b \text{ “×” } (y_1, y_2)) &= (-a(x_1, x_2)) \text{ “+” } (-b(y_1, y_2)) \\ &= (-a(x_1, x_2)) - (-b(x_1, x_2)) \\ &= -a(x_1, x_2) + b(x_1, x_2) \\ &= -1(1, 1) + 2(2, 2) \\ &= (1, 1).\end{aligned}$$

Since we have $(-3, 3) \neq (1, 1)$, we conclude

$$(a + b) \text{ “×” } (x_1, x_2) \neq (a \text{ “×” } (x_1, x_2)) \text{ “+” } (b \text{ “×” } (y_1, y_2)).$$

Therefore, we conclude that $\mathbb{F}^n$ is not a vector space with respect to the specified operations because it does not satisfy all the properties of a vector space. □

(20pts) 3. For each of the following subsets of $\mathbb{F}^3$, determine whether it is a subspace of $\mathbb{F}^3$. If so, prove it. If not, give a counterexample to show some property of a subspace that is not satisfied.

(5pts) a. $\{(x_1, x_2, x_3) \in \mathbb{F}^3 : x_1 + 2x_2 + 3x_3 = 0\}$;

Proof. We will prove that $U_1 = \{(x_1, x_2, x_3) \in \mathbb{F}^3 : x_1 + 2x_2 + 3x_3 = 0\}$ is a subspace of $\mathbb{F}^3$.

- Additive identity: Since $(0) + 2(0) + 3(0) = 0$, we have $(0, 0, 0) \in U_1$.
- Closed under addition: Suppose we have $(x_1, x_2, x_3), (y_1, y_2, y_3) \in U_1$. Then we have $x_1 + 2x_2 + 3x_3 = 0$ and $y_1 + 2y_2 + 3y_3 = 0$. These imply

$$\begin{aligned}(x_1 + y_1) + 2(x_2 + y_2) + 3(x_3 + y_3) &= (x_1 + 2x_2 + 3x_3) + (y_1 + 2y_2 + 3y_3)\\ &= 0 + 0\\ &= 0.\end{aligned}$$

  So we conclude $(x_1, x_2, x_3) + (y_1, y_2, y_3) = (x_1 + y_1, x_2 + y_2, x_3 + y_3) \in U_1$.
- Closed under scalar multiplication: Suppose we have $\lambda \in \mathbb{F}$ and $(x_1, x_2, x_3) \in U_1$. Then we have $x_1 + 2x_2 + 3x_3 = 0$. This implies

$$\begin{aligned}(\lambda x_1) + 2(\lambda x_2) + 3(\lambda x_3) &= \lambda(x_1 + 2x_2 + 3x_3)\\ &= \lambda \cdot 0\\ &= 0.\end{aligned}$$

  So we conclude $\lambda(x_1, x_2, x_3) = (\lambda x_1, \lambda x_2, \lambda x_3) \in U_1$.

Since we satisfied all the properties of a subspace, we conclude that U_1 is a subspace of $\mathbb{F}^3$. □

(5pts) b. $\{(x_1, x_2, x_3) \in \mathbb{F}^3 : x_1 + 2x_2 + 3x_3 = 4\}$;

Proof. We will prove that $U_2 = \{(x_1, x_2, x_3) \in \mathbb{F}^3 : x_1 + 2x_2 + 3x_3 = 4\}$ is not a subspace of $\mathbb{F}^3$.

- Additive identity is not satisfied. Since we have $0 + 2(0) + 3(0) = 0 \neq 4$, we conclude $(0, 0, 0) \notin U_2$.

Since we showed that one of the properties of a subspace is not satisfied, we conclude that U_2 is not a subspace of $\mathbb{F}^3$. □

(5pts) c. $\{(x_1, x_2, x_3) \in \mathbb{F}^3 : x_1x_2x_3 = 0\}$;

Proof. We will prove that $U_3 = \{(x_1, x_2, x_3) \in \mathbb{F}^3 : x_1x_2x_3 = 0\}$ is not a subspace of $\mathbb{F}^3$.

- Closed under addition is not satisfied. Let $(x_1, x_2, x_3) = (0, 1, 0), (y_1, y_2, y_3) = (1, 0, 1) \in \mathbb{F}^3$. Then we have $x_1x_2x_3 = (0)(1)(0)$ and $y_1y_2y_3 = (1)(0)(1) = 0$, which means we have $(x_1, x_2, x_3), (y_1, y_2, y_3) \in U_3$. But these imply

$$\begin{aligned}(x_1 + y_1)(x_2 + y_2)(x_3 + y_3) &= (0 + 1)(1 + 0)(0 + 1)\\ &= (1)(1)(1)\\ &= 1\\ &\neq 0,\end{aligned}$$

  which means we have $(x_1, x_2, x_3) + (y_1, y_2, y_3) = (x_1 + y_1, x_2 + y_2, x_3 + y_3) \notin U_3$.

Since we showed that one of the properties of a subspace is not satisfied, we conclude that U_3 is not a subspace of $\mathbb{F}^3$. □

(5pts) d. $\{(x_1, x_2, x_3) \in \mathbb{F}^3 : x_1 = 5x_3\}$.

Proof. We will prove that $U_4 = \{(x_1, x_2, x_3) \in \mathbb{F}^3 : x_1 = 5x_3\}$ is a subspace of $\mathbb{F}^3$.

- Additive identity: Since $(0) = 5(0)$, we have $(0, 0, 0) \in U_4$.
- Closed under addition: Suppose we have $(x_1, x_2, x_3), (y_1, y_2, y_3) \in U_4$. Then we have $x_1 = 5x_3$ and $y_1 = 5y_3$. These imply

$$\begin{aligned}x_1 + y_1 &= 5x_3 + 5y_3\\ &= 5(x_3 + y_3).\end{aligned}$$

  So we conclude $(x_1, x_2, x_3) + (y_1, y_2, y_3) = (x_1 + y_1, x_2 + y_2, x_3 + y_3) \in U_1$.
- Closed under scalar multiplication: Suppose we have $\lambda \in \mathbb{F}$ and $(x_1, x_2, x_3) \in U_1$. Then we have $x_1 = 5x_3$. This implies

$$\begin{aligned}\lambda x_1 &= \lambda(5x_3)\\ &= 5(\lambda x_3).\end{aligned}$$

  So we conclude $\lambda(x_1, x_2, x_3) = (\lambda x_1, \lambda x_2, \lambda x_3) \in U_1$.

Since we satisfied all the properties of a subspace, we conclude that U_4 is a subspace of $\mathbb{F}^3$. □

(20pts) 4. Let $\mathbb{R}^\mathbb{R}$ be the set of all real-valued functions $f : \mathbb{R} \to \mathbb{R}$. A real-valued function $f : \mathbb{R} \to \mathbb{R}$ is called *even* if

$$f(-x) = f(x)$$

for all $x \in \mathbb{R}$. A real-valued function $f : \mathbb{R} \to \mathbb{R}$ is called *odd* if

$$f(-x) = -f(x)$$

for all $x \in \mathbb{R}$. Let U_e denote the set of real-valued even functions on $\mathbb{R}$, and let U_o denote the set of real-valued odd functions on $\mathbb{R}$. Show that we have $\mathbb{R}^\mathbb{R} = U_e \oplus U_o$.

Proof. First, we need to show that U_e and U_o are subspaces of $\mathbb{R}^\mathbb{R}$.

- Additive identity: For all $x \in \mathbb{R}$, the zero function satisfies $0(x) = 0 = 0(-x)$ and $0(x) = 0 = -0 = -0(x)$. So we have $0 \in U_e$ and $0 \in U_o$.
- Closed under addition: Let $g_1, g_2 \in U_e$ and $h_1, h_2 \in U_o$ be arbitrary. Then, g_1, g_2 are even and h_1, h_2 are odd; in other words, for all $x \in \mathbb{R}$, we have $g_1(x) = g_1(-x)$, $g_2(x) = g_2(-x)$, $h_1(-x) = -h_1(x)$, and $h_2(-x) = -h_2(x)$. So, for all $x \in \mathbb{R}$, we have

$$\begin{aligned}(g_1 + g_2)(-x) &= g_1(-x) + g_2(-x)\\ &= g_1(x) + g_2(x)\\ &= (g_1 + g_2)(x)\end{aligned}$$

  and

$$\begin{aligned}(h_1 + h_2)(-x) &= h_1(-x) + h_2(-x)\\ &= -h_1(x) - h_2(x)\\ &= -(h_1(x) + h_2(x))\\ &= -(h_1 + h_2)(x).\end{aligned}$$

  So $g_1 + g_2$ is even and $h_1 + h_2$ is odd; in other words, we have $g_1 + g_2 \in U_e$ and $h_1 + h_2 \in U_o$.
- Closed under scalar multiplication: Let $\lambda \in \mathbb{F}$, $g \in U_e$, and $h \in U_o$ be arbitrary. Then, g is even and h is odd; in other words, for all $x \in \mathbb{R}$, we have $g(-x) = g(x)$ and $h(-x) = -h(x)$. So, for all $x \in \mathbb{R}$, we have

$$\begin{aligned}(\lambda g)(-x) &= \lambda g(-x)\\ &= \lambda g(x)\\ &= (\lambda g)(x)\end{aligned}$$

  and

$$\begin{aligned}(\lambda h)(-x) &= \lambda h(-x)\\ &= \lambda(-h(x))\\ &= -\lambda h(x)\\ &= (\lambda h)(x).\end{aligned}$$

  So λg is even and λh is odd; in other words, we have $\lambda g \in U_e$ and $\lambda h \in U_o$.

Since we satisfied all the properties of a subspace, we conclude that U_e and U_o are subspaces of $\mathbb{R}^\mathbb{R}$. Next, we need to show $\mathbb{R}^\mathbb{R} = U_e + U_o$. In other words, we will show that we can write every function $f \in \mathbb{R}^\mathbb{R}$ as a sum of an even function and an odd function. Define for all $x \in \mathbb{R}$ the functions $g, h \in \mathbb{R}^\mathbb{R}$ by

$$g(x) = \frac{f(x) + f(-x)}{2}$$

and

$$h(x) = \frac{f(x) - f(-x)}{2}.$$

Then

$$\begin{aligned}g(-x) &= \frac{f(-x) + f(-(-x))}{2}\\ &= \frac{f(-x) + f(x)}{2}\\ &= \frac{f(x) + f(-x)}{2}\\ &= g(x),\end{aligned}$$

which means g is even, or $g \in U_e$. Similarly,

$$\begin{aligned} h(-x) &= \frac{f(-x) - f(-(-x))}{2} \\ &= -\frac{f(x) - f(-x)}{2} \\ &= -h(x), \end{aligned}$$

which means h is odd, or $h \in U_o$. Finally, for all $x \in \mathbb{R}$, we have

$$\begin{aligned} (g + h)(x) &= g(x) + h(x) \\ &= \frac{f(x) + f(-x)}{2} + \frac{f(x) - f(-x)}{2} \\ &= \frac{(f(x) + f(-x)) + (f(x) - f(-x))}{2} \\ &= \frac{2f(x)}{2} \\ &= f(x), \end{aligned}$$

and so $f = g + h$, which establishes $\mathbb{R}^{\mathbb{R}} = U_e + U_o$. At this point, it remains to show $\mathbb{R}^{\mathbb{R}} = U_e \oplus U_o$. According to 1.45 of Axler, we only need to show $U_e \cap U_o = \{0\}$. So suppose we have $f \in U_e \cap U_o$. Then $f \in U_e$ and $f \in U_o$, which means f is both even and odd. In other words, f satisfies both $f(-x) = f(x)$ and $f(-x) = -f(x)$ for all $x \in \mathbb{R}$. Combining the two equations gives us $-f(x) = f(x)$, which implies $f(x) = 0$ for all $x \in \mathbb{R}$. Therefore, $f = 0 \in \{0\}$, and so we have $U_e \cap U_0 \subset \{0\}$. On the other hand, since $U_e \cap U_o$ is a subspace of $\mathbb{R}^{\mathbb{R}}$, we have in fact the set equality $U_e \cap U_0 = \{0\}$. By 1.45 of Axler, we conclude $\mathbb{R}^{\mathbb{R}} = U_e \oplus U_o$. □

(20pts) 5. Let $\mathbb{C}^{\mathbb{R}}$ be the set of all complex-valued functions $f : \mathbb{R} \to \mathbb{C}$. Define on $\mathbb{C}^{\mathbb{R}}$ the usual operations of addition

$$(f + g)(x) = f(x) + g(x)$$

and scalar multiplication

$$(\lambda f)(x) = \lambda f(x)$$

for all scalars $\lambda \in \mathbb{F}$ and complex-valued functions $f, g \in \mathbb{C}^{\mathbb{R}}$. Also let U be the set of all complex-valued functions $f : \mathbb{R} \to \mathbb{C}$ such that

$$f(-x) = \overline{f(x)}$$

for all $x \in \mathbb{R}$, where the bar denotes the complex conjugate.

(12pts) a. Show that $\mathbb{C}^{\mathbb{R}}$ is a vector space over $\mathbb{R}$ with the operations defined above.

Proof. Let $f, g, h \in \mathbb{C}^{\mathbb{R}}$ be arbitrary; this means we will argue for all $f, g, h \in \mathbb{C}^{\mathbb{R}}$.

- Commutativity: For all $x \in \mathbb{R}$, we have

$$\begin{aligned} (f + g)(x) &= f(x) + g(x) \\ &= g(x) + f(x) \\ &= (g + f)(x), \end{aligned}$$

  and so we conclude $f + g = g + f$.
- Associativity: For all $x \in \mathbb{R}$, we have

$$\begin{aligned} ((f + g) + h)(x) &= (f + g)(x) + h(x) \\ &= (f(x) + g(x)) + h(x) \\ &= f(x) + (g(x) + h(x)) \\ &= f(x) + (g + h)(x) \\ &= (f + (g + h))(x), \end{aligned}$$

  and so we conclude $(f + g) + h = f + (g + h)$.
- Additive identity: Suppose we have the zero function $0 \in \mathbb{C}^{\mathbb{R}}$. For all $x \in \mathbb{R}$, we have

$$\begin{aligned} (f + 0)(x) &= f(x) + 0(x) \\ &= f(x) + 0 \\ &= f(x), \end{aligned}$$

  and so we conclude $f + 0 = f$.

- Additive inverse: Suppose we have $-f \in \mathbb{C}^{\mathbb{R}}$. For all $x \in \mathbb{R}$, we have

$$\begin{aligned}(f + (-f))(x) &= f(x) + (-f)(x) \\ &= f(x) - f(x) \\ &= 0,\end{aligned}$$

and so we conclude $f + (-f) = 0$, which means $-f$ is the additive inverse.
- Multiplicative identity: Suppose we have the identity function $1 \in \mathbb{C}^{\mathbb{R}}$. For all $x \in \mathbb{R}$, we have

$$\begin{aligned}(1f)(x) &= 1f(x) \\ &= f(x),\end{aligned}$$

and so we conclude $1f = f$.
- Distributive properties: Let $\lambda \in \mathbb{R}$ be arbitrary. Then, for all $x \in \mathbb{R}$, we have

$$\begin{aligned}(\lambda(f + g))(x) &= \lambda(f + g)(x) \\ &= \lambda(f(x) + g(x)) \\ &= \lambda f(x) + \lambda g(x) \\ &= (\lambda f + \lambda g)(x),\end{aligned}$$

and so we conclude $\lambda(f + g) = \lambda f + \lambda g$.

Since we satisfied all the properties of a vector space, we conclude that $\mathbb{C}^{\mathbb{R}}$ is a vector space. □

(8pts) b. Show that U is a subspace of $\mathbb{C}^{\mathbb{R}}$.

Proof. Let $f, g \in U$ be arbitrary; this means we will be arguing for all $f, g \in U$.

- Additive identity: For all $x \in \mathbb{R}$, the zero function $0 \in \mathbb{C}^{\mathbb{R}}$ satisfies $\overline{0(x)} = \overline{0} = 0 = 0(x)$ for all $x \in \mathbb{R}$, and so we conclude $0 \in U$.
- Closed under addition: Since we assumed $f, g \in U$, we have $f(-x) = \overline{f(x)}$ and $g(-x) = \overline{g(x)}$ for all $x \in \mathbb{R}$. These imply

$$\begin{aligned}(f + g)(-x) &= f(-x) + g(-x) \\ &= \overline{f(x)} + \overline{g(x)} \\ &= \overline{f(x) + g(x)} \\ &= \overline{(f + g)(x)}\end{aligned}$$

for all $x \in \mathbb{R}$, and so we conclude $f + g \in U$.
- Closed under scalar multiplication: Let $\lambda \in \mathbb{R}$ be arbitrary. Since we assumed $f \in U$, we have $f(-x) = \overline{f(x)}$ for all $x \in \mathbb{R}$. This implies

$$\begin{aligned}(\lambda f)(-x) &= \lambda f(-x) \\ &= \lambda \overline{f(x)} \\ &= \overline{\lambda}\overline{f(x)} \\ &= \overline{\lambda f(x)} \\ &= \overline{(\lambda f)(x)}\end{aligned}$$

for all $x \in \mathbb{R}$, and so we conclude $\lambda f \in U$.

Since we satisfied all the properties of a subspace, we conclude that U is a subspace of $\mathbb{C}^{\mathbb{R}}$. □