MATH 131: Linear Algebra I
University of California, Riverside
Group Examination 2 Solutions
July 3, 2019

(20pts) 1. For this question, you will need to refer to the definitions in Chapter 2 of your Axler textbook to find the answers.

(4pts) a. Write down the definitions of *linear combination* and *span*.

Definition. From Definitions 2.3 and 2.5 of Axler:

- A *linear combination* of a list $v_1, \ldots, v_m$ of vectors in V is a vector of the form $a_1 v_1 + \cdots + a_m v_m$ for some $a_1, \ldots, a_m \in \mathbb{F}$.
- The set of all linear combinations of a list of vectors $v_1, \ldots, v_m$ in V is called the *span* of $v_1, \ldots, v_m$, denoted $\operatorname{span}(v_1, \ldots, v_m) = \{a_1 v_1 + \cdots a_m v_m : a_1, \ldots, a_m \in \mathbb{F}\}$.

(4pts) b. Write down the definitions of *linearly independent* and *linearly dependent*.

Definition. From Definitions 2.17 and 2.19 of Axler:

- A list $v_1, \ldots, v_m$ of vectors in V is called *linearly independent* if the only choice of $a_1, \ldots, a_m \in \mathbb{F}$ that satisfies $a_1 v_1 + \cdots + a_m v_m = 0$ is $a_1 = 0, \ldots, a_m = 0$.
- A list $v_1, \ldots, v_m$ of vectors in V is called *linearly dependent* if there exist $a_1, \ldots, a_m \in \mathbb{F}$, not all 0, such that $a_1 v_1 + \cdots + a_m v_m = 0$.

(4pts) c. Write down the definitions of *finite-dimensional vector space* and *infinite-dimensional vector space*.

Definition. From Definitions 2.10 and 2.15 of Axler:

- A vector space is called *finite-dimensional* if some list of vectors $v_1, \ldots, v_m$ in it spans the space; in other words, if we have $\operatorname{span}(v_1, \ldots, v_m) = V$.
- A vector space is called *infinite-dimensional* if it is not finite-dimensional.

(4pts) d. Write down the definitions of *polynomial* and *degree of a polynomial*.

Definition. From Definitions 2.11 and 2.12 of Axler:

- A function $p : \mathbb{F} \to \mathbb{F}$ is called a polynomial with coefficients in $\mathbb{F}$ if there exist $a_0, \ldots, a_m \in \mathbb{F}$ such that $p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_m z^m$ for all $z \in \mathbb{F}$. The set of all polynomials with coefficients in $\mathbb{F}$ is denoted $\mathcal{P}(\mathbb{F})$.
- A polynomial $p \in \mathcal{P}(\mathbb{F})$ is said to have *degree m* if there exist scalars $a_0, a_1, \ldots, a_m \in \mathbb{F}$ with $a_m \neq 0$ such that $p(z) = a_0 + a_1 z + \cdots + a_m z^m$ for all $z \in \mathbb{F}$. If p has degree m, we write $\deg p = m$.

(4pts) e. Write down the definitions of *basis* and *dimension*.

Definition. From Definitions 2.27 and 2.36 of Axler:

- A *basis* of V is a list of vector in V that is linearly independent and spans V.
- The *dimension* of a finite-dimensional vector space is the length in any basis of the vector space. The dimension of of the vector space V is denoted by $\dim V$.

(20pts) 2. Suppose $v_1, v_2, v_3, \ldots, v_m$ is a linearly independent list of vectors in the vector space V.

(8pts) a. Prove that $5v_1 - 4v_2, v_2, v_3, \ldots, v_m$ is linearly independent.

Proof. Suppose $a_1, \ldots, a_m \in \mathbb{F}$ satisfy

$$a_1(5v_1 - 4v_2) + a_2 v_2 + a_3 v_3 + \cdots + a_m v_m = 0.$$

Algebraically rearranging the left-hand side of the above equation gives

$$(5a_1)v_1 + (-4a_1 + a_2)v_2 + a_3 v_3 + \cdots + a_m v_m = 0.$$

Since $v_1, v_2, v_3, \ldots, v_m$ is linearly independent, all scalars are zero, which means we have

$$5a_1 = 0, -4a_1 + a_2 = 0, a_3 = 0, \ldots, a_m = 0.$$

The first equation $5a_1 = 0$ implies $a_1 = 0$. The second equation $-4a_1 + a_2 = 0$ with $a_1 = 0$ implies $a_2 = 0$. So we have

$$a_1 = 0, a_2 = 0, a_3 = 0, \ldots, a_m = 0,$$

and so we conclude that $5v_1 - 4v_2, v_2, v_3, \ldots, v_m$ is linearly independent. □

(8pts) b. If $\lambda \in \mathbb{F}$ satisfies $\lambda \neq 0$, prove that $\lambda v_1, \lambda v_2, \lambda v_3, \ldots, \lambda v_m$ is linearly independent.

Proof. Suppose $a_1, \ldots, a_m \in \mathbb{F}$ satisfy

$$a_1(\lambda v_1) + \cdots + a_m(\lambda v_m) = 0.$$

Rewriting the parentheses on the left-hand side of the above equation gives

$$(a_1\lambda)v_1 + \cdots + (a_m\lambda)v_m = 0.$$

Since $v_1, \ldots, v_m$ is linearly independent, all scalars are zero, which means we have

$$a_1\lambda = 0, \ldots, a_m\lambda = 0.$$

Because we assumed $\lambda \neq 0$, we arrive at

$$a_1 = 0, \ldots, a_m = 0,$$

and so we conclude that $\lambda v_1, \lambda v_2, \lambda v_3, \ldots, \lambda v_m$ is linearly independent. □

(4pts) c. Assume that $w_1, \ldots, w_m$ is also a linearly independent list of vectors in V. Give a counterexample to show that the list $v_1 + w_1, \ldots, v_m + w_m$ is not linearly independent.

Proof. This is a false statement; we will give a counterexample. Let $m = 2$, let $V = \mathbb{R}^2$, let $v_1 = (1, 0), v_2 = (0, 1)$ be a list of vectors in $\mathbb{R}^2$, and let $w_1 = -v_1 = (-1, 0)$ and $w_2 = -v_2 = (0, -1)$. Suppose a_1, a_2 satisfy

$$a_1 v_1 + a_2 v_2 = (0, 0).$$

Then we have

$$\begin{aligned}(0,0) &= a_1 v_1 + a_2 v_2\\ &= a_1(1,0) + a_2(0,1)\\ &= (a_1, 0) + (0, a_2)\\ &= (a_1, a_2),\end{aligned}$$

from which we get $a_1 = 0, a_2 = 0$, and so v_1, v_2 is linearly independent. Similarly, we have

$$\begin{aligned}(0,0) &= b_1 w_1 + b_2 w_2\\ &= b_1(-1,0) + b_2(0,-1)\\ &= (-b_1, 0) + (0, -b_2)\\ &= (-b_1, -b_2),\end{aligned}$$

from which we get $-b_1 = 0, -b_2 = 0$, or equivalently $b_1 = 0, b_2 = 0$, and so w_1, w_2 is linearly independent. However, if we choose $c_1 = 1, c_2 = 1$, then we have

$$\begin{aligned}c_1(v_1 + w_1) + c_2(v_2 + w_2) &= 1((1,0) + (-1,0)) + 1((0,1) + (0,-1))\\ &= 1(0,0) + 1(0,0)\\ &= (0,0) + (0,0)\\ &= (0,0),\end{aligned}$$

which means $v_1 + w_1, v_2 + w_2$ is not linearly independent. □

(20pts) 3. Consider the vector space $\mathbb{F}^3$ with the standard basis $(1, 0, 0), (0, 1, 0), (0, 0, 1)$.

(15pts) a. Show that the list $(1, 0, -1), (1, 2, 1), (0, -3, 2)$ is a basis of $\mathbb{F}^3$.

Proof. First, we need to show that $(1, 0, -1), (1, 2, 1), (0, -3, 2)$ is a linearly independent set that spans $\mathbb{R}^3$. To do this, suppose $a_1, a_2, a_3 \in \mathbb{F}$ satisfy

$$a_1(1, 0, -1) + a_2(1, 2, 1) + a_3(0, -3, 2) = (0, 0, 0).$$

Then we have

$$\begin{aligned}(0,0,0) &= a_1(1,0,-1) + a_2(1,2,1) + a_3(0,-3,2)\\ &= (a_1, 0, -a_1) + (a_2, 2a_2, a_2) + (0, -3a_3, 2a_3)\\ &= (a_1 + a_2, 2a_2 - 3a_3, -a_1 + a_2 + 2a_3).\end{aligned}$$

Equating the coordinates gives us the system of equations

$$\begin{aligned}a_1 + a_2 &= 0,\\ 2a_2 - 3a_3 &= 0,\\ -a_1 + a_2 + 2a_3 &= 0,\end{aligned}$$

from which system-solving gives $a_1 = 0, a_2 = 0, a_3 = 0$. Therefore, $(1, 0, -1), (1, 2, 1), (0, -3, 2)$ is a linearly independent list. Furthermore, since $(1, 0, -1), (1, 2, 1), (0, -3, 2)$ has length 3 and we have $\dim \mathbb{R}^3 = 3$, it is of the right length, which means, by 2.39 of Axler, this list is a basis of $\mathbb{R}^3$. □

Alternate proof. First, we will prove that the list $(1,0,-1),(1,2,1),(0,-3,2)$ spans $\mathbb{R}^3$. This means we need to show that, for all vectors $(x_1,x_2,x_3)\in\mathbb{R}^3$, there exist $a_1,a_3,a_3\in\mathbb{F}$ such that

$$(x_1,x_2,x_3)=a_1(1,0,-1)+a_2(1,2,1)+a_3(0,-3,2).$$

With that said, we have

$$\begin{aligned}(x_1,x_2,x_3)&=a_1(1,0,-1)+a_2(1,2,1)+a_3(0,-3,2)\\&=(a_1,0,-a_1)+(a_2,2a_2,a_2)+(0,-3a_3,2a_3)\\&=(a_1+a_2,2a_2-3a_3,-a_1+a_2+2a_3).\end{aligned}$$

Equating the coordinates gives us the system of equations

$$\begin{aligned}a_1+a_2&=x_1,\\2a_2-3a_3&=x_2,\\-a_1+a_2+2a_3&=x_3,\end{aligned}$$

from which system-solving gives

$$\begin{aligned}a_1&=\frac{7}{10}x_1-\frac{1}{5}x_2-\frac{3}{10}x_3,\\a_2&=\frac{3}{10}x_1+\frac{1}{5}x_2+\frac{3}{10}x_3,\\a_3&=\frac{1}{5}x_1-\frac{1}{5}x_2+\frac{1}{5}x_3.\end{aligned}$$

So we found $a_1,a_2,a_3\in\mathbb{F}$ that depend on the coordinates of the $(x_1,x_2,x_3)\in\mathbb{R}^3$, which means $(1,0,-1),(1,2,1),(0,-3,2)$ spans $\mathbb{R}^3$. Next, we need to prove the list $(1,0,-1),(1,2,1),(0,-3,2)$ is linearly independent. To do this, suppose $a_1,a_2,a_3\in\mathbb{F}$ satisfy

$$a_1(1,0,-1)+a_2(1,2,1)+a_3(0,-3,2)=(0,0,0).$$

The above equation is only

$$(x_1,x_2,x_3)=a_1(1,0,-1)+a_2(1,2,1)+a_3(0,-3,2)$$

with $x_1=0,x_2=0,x_3=0$. Substituting $x_1=0,x_2=0,x_3=0$ into our expressions of the scalars a_1,a_2,a_3, we get

$$\begin{aligned}a_1&=\frac{7}{10}(0)-\frac{1}{5}(0)-\frac{3}{10}(0)=0,\\a_2&=\frac{3}{10}(0)+\frac{1}{5}(0)+\frac{3}{10}(0)=0,\\a_3&=\frac{1}{5}(0)-\frac{1}{5}(0)+\frac{1}{5}(0)=0.\end{aligned}$$

So the list $(1,0,-1),(1,2,1),(0,-3,2)$ is linearly independent. Therefore, $(1,0,-1),(1,2,1),(0,-3,2)$ is a basis of $\mathbb{R}^3$. □

(5pts) b. Express the standard basis vectors $(1,0,0),(0,1,0),(0,0,1)$ as a linear combination of the list in part (a).

Proof. Let us write each vector $(1,0,0),(0,1,0),(0,0,1)$ as a linear combination of the list $(1,0,-1),(1,2,1),(0,-3,2)$. In other words, we need to find $a_1,a_2,a_3,b_1,b_2,b_3,c_1,c_2,c_3\in\mathbb{F}$ such that

$$\begin{aligned}(1,0,0)&=a_1(1,0,-1)+a_2(1,2,1)+a_3(0,-3,2),\\(0,1,0)&=b_1(1,0,-1)+b_2(1,2,1)+b_3(0,-3,2),\\(0,0,1)&=c_1(1,0,-1)+c_2(1,2,1)+c_3(0,-3,2).\end{aligned}$$

System-solving the first equation as done exactly in part (a) and substituting $x_1=1,x_2=0,x_3=0$, we obtain

$$\begin{aligned}a_1&=\frac{7}{10}(1)-\frac{1}{5}(0)-\frac{3}{10}(0)=\frac{7}{10},\\a_2&=\frac{3}{10}(1)+\frac{1}{5}(0)+\frac{3}{10}(0)=\frac{3}{10},\\a_3&=\frac{1}{5}(1)-\frac{1}{5}(0)+\frac{1}{5}(0)=\frac{1}{5}.\end{aligned}$$

System-solving the second equation as done exactly in part (a) and substituting $x_1=0,x_2=1,x_3=0$, we obtain

$$\begin{aligned}b_1&=\frac{7}{10}(0)-\frac{1}{5}(1)-\frac{3}{10}(0)=-\frac{1}{5},\\b_2&=\frac{3}{10}(0)+\frac{1}{5}(1)+\frac{3}{10}(0)=\frac{1}{5},\\b_3&=\frac{1}{5}(0)-\frac{1}{5}(1)+\frac{1}{5}(0)=-\frac{1}{5}.\end{aligned}$$

System-solving the third equation as done exactly in part (a) and substituting $x_1 = 0, x_2 = 0, x_3 = 1$, we obtain

$$\begin{aligned} c_1 &= \frac{7}{10}(0) - \frac{1}{5}(0) - \frac{3}{10}(1) = -\frac{3}{10}, \\ c_2 &= \frac{3}{10}(0) + \frac{1}{5}(0) + \frac{3}{10}(1) = \frac{3}{10}, \\ c_3 &= \frac{1}{5}(0) - \frac{1}{5}(0) + \frac{1}{5}(1) = \frac{1}{5}. \end{aligned}$$

This completes our proof. □

(20pts) 4. Let V be a vector space. Suppose v_1, v_2, v_3, v_4 is a basis of V. Prove that the list

$$v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4$$

is also a basis of V.

Proof. Suppose $a_1, a_2, a_3, a_4 \in \mathbb{F}$ satisfy

$$a_1(v_1 - v_2) + a_2(v_2 - v_3) + a_3(v_3 - v_4) + a_4 v_4 = 0.$$

Algebraically rearranging the terms, we get

$$a_1 v_1 + (-a_1 + a_2)v_2 + (-a_2 + a_3)v_3 + (-a_3 + a_4)v_4 = 0.$$

Since v_1, v_2, v_3, v_4 is linearly independent, all scalars are zero, which means we have

$$a_1 = 0, -a_1 + a_2 = 0, -a_2 + a_3 = 0, -a_3 + a_4 = 0.$$

The second equation $-a_1 + a_2 = 0$ with $a_1 = 0$ implies $a_2 = 0$. The third equation $-a_2 + a_3 = 0$ with $a_2 = 0$ implies $a_3 = 0$. The fourth equation $-a_3 + a_4 = 0$ with $a_3 = 0$ implies $a_4 = 0$. So we have

$$a_1 = 0, a_2 = 0, a_3 = 0, a_4 = 0,$$

and so we conclude $v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4$ is linearly independent. Next, we need to prove that $v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4$ spans V. Since v_1, v_2, v_3, v_4 spans V, there exist $a_1, a_2, a_3, a_4 \in \mathbb{F}$ such that

$$v = a_1 v_1 + a_2 v_2 + a_3 v_3 + a_4 v_4.$$

Furthermore, observe that we can write

$$\begin{aligned} v_1 &= (v_1 - v_2) + (v_2 - v_3) + (v_3 - v_4) + v_4, \\ v_2 &= (v_2 - v_3) + (v_3 - v_4) + v_4, \\ v_3 &= (v_3 - v_4) + v_4. \end{aligned}$$

So we have

$$\begin{aligned} v &= a_1 v_1 + a_2 v_2 + a_3 v_3 + a_4 v_4 \\ &= a_1((v_1 - v_2) + (v_2 - v_3) + (v_3 - v_4) + v_4) + a_2((v_2 - v_3) + (v_3 - v_4) + v_4) + a_3((v_3 - v_4) + v_4) + a_4 v_4 \\ &= a_1(v_1 - v_2) + (a_1 + a_2)(v_2 - v_3) + (a_1 + a_2 + a_3)(v_3 - v_4) + (a_1 + a_2 + a_3 + a_4)v_4. \end{aligned}$$

Since we also have $a_1, a_1 + a_2, a_1 + a_2 + a_3, a_1 + a_2 + a_3 + a_4 \in \mathbb{F}$, it follows that the list $v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4$ spans V. Therefore, $v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4$ is a basis of V. □

(20pts) 5. Consider the sets

$$U_1 = \{(x_1, x_2, x_3, x_4) \in \mathbb{F}^4 : x_1 + x_2 = 0\}$$

and

$$U_2 = \{(x_1, x_2, x_3, x_4) \in \mathbb{F}^4 : x_1 + x_3 = 0\}.$$

(6pts) a. Show that U_1 and U_2 are subspaces of $\mathbb{F}^4$.

Proof. We will prove that $U_1 = \{(x_1, x_2, x_3, x_4) \in \mathbb{F}^4 : x_1 + x_2 = 0\}$ and $U_2 = \{(x_1, x_2, x_3, x_4) \in \mathbb{F}^4 : x_1 + x_3 = 0\}$ are subspaces of $\mathbb{F}^4$.

- Additive identity: Since $(0) + (0) = 0$, we have $(0, 0, 0, 0) \in U_1$ and $(0, 0, 0, 0) \in U_2$. Both U_1 and U_2 contain the same additive identity.

- Closed under addition: Suppose we have $(x_1, x_2, x_3, x_4), (y_1, y_2, y_3, y_4) \in U_1$. Then $x_1 + x_2 = 0$ and $y_1 + y_2 = 0$. So we have

$$\begin{aligned}(x_1 + y_1) + (x_2 + y_2) &= x_1 + x_2 + y_1 + y_2 \\ &= 0 + 0 \\ &= 0.\end{aligned}$$

So we conclude $(x_1, x_2, x_3, x_4) + (y_1, y_2, y_3, y_4) = (x_1 + y_1, x_2 + y_2, x_3 + y_3, x_4 + y_4) \in U_1$, and so U_1 is closed under addition. Similarly, suppose we have $(z_1, z_2, z_3, z_4), (w_1, w_2, w_3, w_4) \in U_1$. Then $x_1 + x_3 = 0$ and $y_1 + y_3 = 0$. So we have

$$\begin{aligned}(z_1 + w_1) + (z_3 + w_3) &= z_1 + z_3 + w_1 + w_3 \\ &= 0 + 0 \\ &= 0.\end{aligned}$$

So we conclude $(z_1, z_2, z_3, z_4) + (w_1, w_2, w_3, w_4) = (z_1 + w_1, z_2 + w_2, z_3 + w_3, z_4 + w_4) \in U_2$, and so U_2 is closed under addition.
- Closed under scalar multiplication: Suppose we have $\lambda \in \mathbb{F}$ and $(x_1, x_2, x_3) \in U_1$. Then we have $x_1 + x_2 = 0$. This implies

$$\begin{aligned}(\lambda x_1) + (\lambda x_2) &= \lambda(x_1 + x_2) \\ &= \lambda \cdot 0 \\ &= 0.\end{aligned}$$

So we conclude $\lambda(x_1, x_2, x_3, x_4) = (\lambda x_1, \lambda x_2, \lambda x_3, \lambda x_4) \in U_1$, and so U_1 is closed under scalar multiplication. Similarly, suppose we have $\lambda \in \mathbb{F}$ and $(z_1, z_2, z_3) \in U_1$. Then we have $z_1 + z_3 = 0$. This implies

$$\begin{aligned}(\lambda z_1) + (\lambda z_3) &= \lambda(z_1 + z_3) \\ &= \lambda \cdot 0 \\ &= 0.\end{aligned}$$

So we conclude $\lambda(z_1, z_2, z_3, z_4) = (\lambda z_1, \lambda z_2, \lambda z_3, \lambda z_4) \in U_2$, and so U_2 is closed under scalar multiplication.

Since we satisfied all the properties of a subspace, we conclude that U_1 and U_2 are subspaces of $\mathbb{F}^4$. □

(12pts) b. Find the dimensions of U_1, U_2, $U_1 + U_2$, and $U_1 \cap U_2$. For this part of the question, do NOT use the formula provided by 2.43 of Axler to find the dimension of $U_1 + U_2$; any justification using that formula will receive an automatic zero score for part (b).

Hint: Find a basis of each of the three sets, and prove that they are indeed bases of their respective sets. What is the length of each basis?

Proof. First, we will work with U_1. Let $(x_1, x_2, x_3, x_4) \in U_1$ be arbitrary. Then we have $x_1 + x_2 = 0$, and so we can write

$$\begin{aligned}(x_1, x_2, x_3, x_4) &= (x_1, -x_1, x_3, x_4) \\ &= (x_1, -x_1, 0, 0) + (0, 0, x_3, 0) + (0, 0, 0, x_4) \\ &= x_1(1, -1, 0, 0) + x_3(0, 0, 1, 0) + x_4(0, 0, 0, 1)\end{aligned}$$

which means the list $(1, -1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1)$ spans U_1. Also, suppose $x_1, x_3, x_4 \in \mathbb{F}$ satisfy

$$x_1(1, -1, 0, 0) + x_3(0, 0, 1, 0) + x_4(0, 0, 0, 1) = (0, 0, 0, 0).$$

The equation then becomes

$$(x_1, -x_1, x_3, x_4) = (0, 0, 0, 0),$$

from which we get $x_1 = 0, x_3 = 0, x_4 = 0$. So the list $(1, -1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1)$ is also linearly independent. Therefore, $(1, -1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1)$ is a basis of U_1. Since the basis of U_1 has length 3, we have $\dim U_1 = 3$. Next, we will work with U_2. Let $(x_1, x_2, x_3, x_4) \in U_2$ be arbitrary. Then we have $x_1 + x_3 = 0$, and so we can write

$$\begin{aligned}(x_1, x_2, x_3, x_4) &= (x_1, x_2, -x_1, x_4) \\ &= (x_1, 0, -x_1, 0) + (0, x_2, 0, 0) + (0, 0, 0, x_4) \\ &= x_1(1, 0, -1, 0) + x_2(0, 0, 1, 0) + x_4(0, 0, 0, 1),\end{aligned}$$

which means the list $(1, 0, -1, 0), (0, 0, 1, 0), (0, 0, 0, 1)$ spans U_2. Also, suppose $x_1, x_3, x_4 \in \mathbb{F}$ satisfy

$$x_1(1, 0, -1, 0) + x_2(0, 0, 1, 0) + x_4(0, 0, 0, 1) = (0, 0, 0, 0).$$

The equation then becomes
$$(x_1, x_2, -x_1, x_4) = (0, 0, 0, 0),$$
from which we get $x_1 = 0, x_2 = 0, x_4 = 0$. So the list $(1, 0, -1, 0), (0, 0, 1, 0), (0, 0, 0, 1)$ is also linearly independent. Therefore, $(1, 0, -1, 0), (0, 0, 1, 0), (0, 0, 0, 1)$ is a basis of U_2. Since the basis of U_2 has length 3, we have $\dim U_2 = 3$. Finally, we will work with

$$U_1 \cap U_2 = \{(x_1, x_2, x_3, x_4) \in \mathbb{F}^4 : x_1 + x_2 = 0 \text{ and } x_1 + x_3 = 0\}.$$

Let $(x_1, x_2, x_3, x_4) \in U_1 \cap U_2$ be arbitrary. Then we have $x_1 + x_2 = 0$ and $x_1 + x_3 = 0$, and so we can write

$$\begin{aligned}(x_1, x_2, x_3, x_4) &= (x_1, -x_1, -x_1, x_4)\\ &= (x_1, -x_1, -x_1, 0) + (0, 0, 0, x_4)\\ &= x_1(1, -1, -1, 0) + x_4(0, 0, 0, 1),\end{aligned}$$

which means the list $(1, -1, -1, 0), (0, 0, 0, 1)$ spans U_1. Also, suppose $x_1, x_3, x_4 \in \mathbb{F}$ satisfy

$$x_1(1, -1, -1, 0) + x_4(0, 0, 0, 1) = (0, 0, 0, 0).$$

The equation then becomes
$$(x_1, -x_1, -x_1, x_4) = (0, 0, 0, 0),$$
from which we get $x_1 = 0, x_4 = 0$. So the list $(1, -1, -1, 0), (0, 0, 0, 1)$ is also linearly independent. Therefore, $(1, -1, -1, 0), (0, 0, 0, 1)$ is a basis of $U_1 \cap U_2$. Since the basis of $U_1 \cap U_2$ contains 3 elements, we have $\dim(U_1 \cap U_2) = 3$. Finally, consider the elements in either the basis of U_1 or the basis of U_2:

$$(1, -1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1), (1, 0, -1, 0), (0, 0, 1, 0), (0, 0, 0, 1).$$

Since $(0, 0, 1, 0)$ and $(0, 0, 0, 1)$ is printed twice, we can delete the vectors from the above list to write down the reduced list
$$(1, -1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1), (1, 0, -1, 0).$$
We will show that this reduced list is a basis of $U_1 + U_2$. Let $(x_1, x_2, x_3, x_4) + (y_1, y_2, y_3, y_4) \in U_1 + U_2$ be an arbitrary vector. Then we have $(x_1, x_2, x_3, x_4) \in U_1, (y_1, y_2, y_3, y_4) \in U_2$, which means the points satisfy $x_1 + x_2 = 0$ and $y_1 + y_3 = 0$. So we have

$$\begin{aligned}(x_1, x_2, x_3, x_4) + (y_1, y_2, y_3, y_4) &= (x_1, -x_1, x_3, x_4) + (y_1, y_2, -y_1, y_4)\\ &= ((x_1, -x_1, 0, 0) + (0, 0, x_3, 0) + (0, 0, 0, x_4)) + ((y_1, 0, -y_1, 0) + (0, y_2, 0, 0) + (0, 0, 0, y_4))\\ &= x_1(1, -1, 0, 0) + x_3(0, 0, 1, 0) + x_4(0, 0, 0, 1) + y_1(1, 0, -1, 0) + y_2(0, 1, 0, 0) + y_4(0, 0, 0, 1)\\ &= x_1(1, -1, 0, 0) + (x_3 + y_2)(0, 1, 0, 0) + (x_4 + y_4)(0, 0, 0, 1).\end{aligned}$$

Since we have $x_1, x_3 + y_2, x_4 + y_4 \in \mathbb{F}$, we can write every $(x_1, x_2, x_3, x_4) + (y_1, y_2, y_3, y_4) \in U_1 + U_2$ as a linear combination of the list
$$(1, -1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1), (1, 0, -1, 0).$$
Now we need to prove that this list is linearly independent. Suppose $a_1, a_2, a_3, a_4 \in \mathbb{F}$ satisfy

$$a_1(1, -1, 0, 0) + a_2(0, 0, 1, 0) + a_3(0, 0, 0, 1) + a_4(1, 0, -1, 0) = (0, 0, 0, 0).$$

Applying addition and scalar multiplication of vectors in $\mathbb{F}$ to the left-hand side of the above equation, we get

$$(a_1 + a_4, -a_1, a_2 - a_4, a_3) = (0, 0, 0, 0),$$

So from which we can equate the coordinates to get

$$a_1 + a_4 = 0, -a_1 = 0, a_2 - a_4 = 0, a_3 = 0.$$

The second equation $-a_1 = 0$ implies $a_1 = 0$. The first equation $a_1 + a_4 = 0$ with $a_1 = 0$ implies $a_4 = 0$. The third equation $a_2 - a_4 = 0$ with $a_4 = 0$ implies $a_2 = 0$. So we have

$$a_1 = 0, a_2 = 0, a_3 = 0, a_4 = 0,$$

and so the list $(1, -1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1), (1, 0, -1, 0)$ is linearly independent. So this list is linearly independent and spans $U_1 + U_2$, which means $(1, -1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1), (1, 0, -1, 0)$ is a basis of $U_1 + U_2$, and so we have $\dim(U_1 + U_2) = 4$. □

(2pts) c. Write down the formula from 2.43 of Axler that represents the dimension of $U_1 + U_2$. Substitute the values you obtained in part (b) into the formula to verify that it holds true for our subspaces U_1 and U_2.

Proof. The formula for the dimension of the sum of U_1 and U_2 is

$$\dim(U_1 + U_2) = \dim U_1 + \dim U_2 - \dim(U_1 \cap U_2).$$

According to our answers to part (b), we have $\dim(U_1 + U_2) = 4$, $\dim U_1 = \dim U_2 = 3$, and $\dim(U_1 \cap U_2) = 2$. So we have

$$\begin{aligned} 4 &= \dim(U_1 + U_2) \\ &= \dim U_1 + \dim U_2 - \dim(U_1 \cap U_2) \\ &= 3 + 3 - 2, \end{aligned}$$

which is a true statement. □