MATH 131: Linear Algebra I
University of California, Riverside
Group Examination 3 Solutions
July 11, 2019

(20pts) 1. For this question, you will need to refer to the definitions in Chapter 3, Sections A-B, of your Axler textbook to find the answers.

(4pts) a. Write down the properties of a *linear map* from V to W.

Properties. From Definition 3.2 of Axler: A *linear map* from V to W is a function $T : V \longrightarrow W$ is the following properties:

- Additivity: $T(u + v) = Tu + Tv$ for all $u, v \in V$;
- Homogeneity: $T(\lambda v) = \lambda(Tv)$ for all $\lambda \in \mathbb{F}$ and for all $v \in V$.

The set of all linear maps from V to W is denoted $\mathcal{L}(V, W)$.

(4pts) b. Write down the differentiation and integration maps, and prove that they are linear.

Definition. From Example 3.4 of Axler:

- Differentiation: Define $D \in \mathcal{L}(\mathcal{P}(\mathbb{R}), \mathcal{P}(\mathbb{R}))$ by $Dp = p'$. The map D is linear because it satisfies
  - Additivity: $D(p + q) = (p + q)' = p' + q' = Dp + Dq$ for all $p, q \in \mathcal{P}(\mathbb{R})$;
  - Homogeneity: $D(\lambda p) = (\lambda p)' = \lambda p' = \lambda Dp$ for all $\lambda \in \mathbb{F}$ and for all $p \in \mathcal{P}(\mathbb{R})$.
- Integration: Define $T \in \mathcal{L}(\mathcal{P}(\mathbb{R}), \mathbb{R})$ by $Tp = \int_0^1 p(x)\,dx$. The map T is linear because it satisfies
  - Additivity: $T(p + q) = \int_0^1 (p(x) + q(x))\,dx = \int_0^1 p(x)\,dx + \int_0^1 q(x)\,dx = Tp + Tq$ for all $p, q \in \mathcal{P}(\mathbb{R})$;
  - Homogeneity: $T(\lambda p) = \int_0^1 (\lambda p)(x)\,dx = \int_0^1 \lambda p(x)\,dx = \lambda \int_0^1 p(x)\,dx = \lambda Tp$ for all $\lambda \in \mathbb{F}$ and for all $p \in \mathcal{P}(\mathbb{R})$.

(4pts) b. Write down the definitions of *addition* and *scalar multiplication* on $\mathcal{L}(V, W)$.

Definition. From Definition 3.6 of Axler: Suppose $S, T \in \mathcal{L}(V, W)$ and $\lambda \in \mathbb{F}$.

- The *sum* $S + T : V \longrightarrow W$ are the linear maps defined by $(S + T)v = Sv + Tv$ for all $v \in V$.
- The *product* (: $V \longrightarrow W$ are the linear maps defined by $(\lambda T)v = \lambda(Tv)$ for all $v \in V$.

(4pts) c. Write down the definitions of *null space* and *range*.

Definition. From Definitions 3.12 and 3.17 of Axler:

- For all $T \in \mathcal{L}(V, W)$, the *null space* of T is the subset of V consisting of those vectors that T maps to 0: $\operatorname{null} T = \{v \in V : Tv = 0\}$.
- For all $T \in \mathcal{L}(V, W)$, the *range* of T is the subset of W consisting of those vectors that are of the form Tv for some $v \in V$: $\operatorname{range} T = \{Tv : v \in V\}$.

(4pts) d. Write down the definitions of *injective* and *surjective*.

Definition. From Definitions 3.15 and 3.20 of Axler:

- A function $T : V \longrightarrow W$ is called *injective* if $Tu = Tv$ implies $u = v$.
- A function $T : V \longrightarrow W$ is called *surjective* if its range equals W; that is, if we have $\operatorname{range} T = W$.

(20pts) 2. Which of the following maps $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is linear? If the map is linear, prove it. If not, give a counterexample to show that some property of a linear map that is not satisfied.

(4pts) a. $T(x_1, x_2) = (x_1 + 1, x_2)$

Proof. We will prove that this map is not linear.

- Homogeneity is not satisfied. Let $\lambda = 2 \in \mathbb{F}$ and let $(x_1, x_2) = (1, 1) \in \mathbb{R}^2$. Then

$$\begin{aligned} T(\lambda(x_1, x_2)) &= T(2(1, 1)) \\ &= T(2, 2) \\ &= (2 + 1, 2) \\ &= (3, 2) \end{aligned}$$

and

$$\begin{aligned} \lambda T(x_1, x_2) &= 2T(1, 1) \\ &= 2(1 + 1, 1) \\ &= 2(2, 1) \\ &= (4, 2). \end{aligned}$$

Since we have $(3, 2) \neq (4, 2)$, we conclude $T(\lambda(x_1, x_2)) \neq \lambda T(x_1, x_2)$.

Since one of the two properties of a linear map are not satisfied, this map is not linear. □

(4pts) b. $T(x_1, x_2) = (x_2, x_1)$

Proof. We will prove that this map is linear.

- Additivity: For all $(x_1, x_2), (y_1, y_2) \in \mathbb{R}^2$, we have

$$\begin{aligned} T((x_1, x_2) + (y_1, y_2)) &= T(x_1 + y_1, x_2 + y_2) \\ &= (x_2 + y_2, x_1 + y_1) \\ &= (x_2, x_1) + (y_2, y_1) \\ &= T(x_1, x_2) + T(y_1, y_2). \end{aligned}$$

- Homogeneity: For all $\lambda \in \mathbb{F}$ and for all $(x_1, x_2) \in \mathbb{R}^2$, we have

$$\begin{aligned} T(\lambda(x_1, x_2)) &= T(\lambda x_1, \lambda x_2) \\ &= (\lambda x_2, \lambda x_1) \\ &= \lambda(x_2, x_1) \\ &= \lambda T(x_1, x_2). \end{aligned}$$

Since the two properties of a linear map are satisfied, this map is linear. □

(4pts) c. $T(x_1, x_2) = (|x_1|, x_2)$

Proof. We will prove that this map is not linear.

- Homogeneity is not satisfied. Let $\lambda = -1 \in \mathbb{F}$ and let $(x_1, x_2) = (1, 1) \in \mathbb{R}^2$. Then

$$\begin{aligned} T(\lambda(x_1, x_2)) &= T(-1(1, 1)) \\ &= T(-1, -1) \\ &= (|-1|, -1) \\ &= (1, -1) \end{aligned}$$

  and

$$\begin{aligned} \lambda T(x_1, x_2) &= -1T(1, 1) \\ &= -1(|1|, 1) \\ &= -1(1, 1) \\ &= (-1, -1). \end{aligned}$$

  Since we have $(1, -1) \neq (-1, -1)$, we conclude $T(\lambda(x_1, x_2)) \neq \lambda T(x_1, x_2)$.

Since one of the two properties of a linear map are not satisfied, this map is not linear. □

(4pts) d. $T(x_1, x_2) = (\sin x_1, x_2)$

Proof. We will prove that this map is not linear.

- Additivity is not satisfied. Let $(x_1, x_2) = (\frac{\pi}{2}, 0), (y_1, y_2) = (\frac{\pi}{2}, 1) \in \mathbb{R}^2$. Then

$$\begin{aligned} T((x_1, x_2) + (y_1, y_2)) &= T\left(\left(\frac{\pi}{2}, 0\right) + \left(\frac{\pi}{2}, 1\right)\right) \\ &= T(\pi, 1) \\ &= (\sin \pi, 1) \\ &= (0, 1) \end{aligned}$$

  and

$$\begin{aligned} T(x_1, x_2) + T(y_1, y_2) &= T\left(\frac{\pi}{2}, 0\right) + T\left(\frac{\pi}{2}, 1\right) \\ &= \left(\sin \frac{\pi}{2}, 0\right) + \left(\sin \frac{\pi}{2}, 1\right) \\ &= (1, 0) + (1, 1) \\ &= (2, 1). \end{aligned}$$

  Since we have $(0, 1) \neq (2, 1)$, we conclude $T(\lambda(x_1, x_2)) \neq \lambda T(x_1, x_2)$.

Since one of the two properties of a linear map are not satisfied, this map is not linear. □

(4pts) e. $T(x_1, x_2) = (x_1 - x_2, 0)$

Proof. We will prove that this map is linear.

- Additivity: For all $(x_1, x_2), (y_1, y_2) \in \mathbb{R}^2$, we have

$$\begin{aligned} T((x_1, x_2) + (y_1, y_2)) &= T(x_1 + y_1, x_2 + y_2) \\ &= ((x_1 + y_1) - (x_2 + y_2), 0) \\ &= (x_1 - x_2 + y_1 - y_2, 0) \\ &= (x_1 - x_2, 0) + (y_1 - y_2, 0) \\ &= T(x_1, x_2) + T(y_1, y_2). \end{aligned}$$

- Homogeneity: For all $\lambda \in \mathbb{F}$ and for all $(x_1, x_2) \in \mathbb{R}^2$, we have

$$\begin{aligned} T(\lambda(x_1, x_2)) &= T(\lambda x_1, \lambda x_2) \\ &= (\lambda x_1 - \lambda x_2, 0) \\ &= (\lambda(x_1 - x_2), 0) \\ &= \lambda(x_1 - x_2, 0) \\ &= \lambda T(x_1, x_2). \end{aligned}$$

Since the two properties of a linear map are satisfied, this map is linear. □

(20pts) 3. Consider a linear map $T : \mathbb{R}^3 \to \mathbb{R}^2$ that satisfies $T(1, -1, 1) = (1, 0)$ and $T(1, 1, 1) = (0, 1)$.

(12pts) a. Show that there exists such a map.
Hint: First verify that $(1, -1, 1), (1, 1, 1)$ is a linearly independent list of vectors in $\mathbb{R}^3$. Then use 2.33 of Axler to show that this linearly independent list extends to a basis of $\mathbb{R}^3$. Finally, use 3.5 of Axler to arrive at your desired conclusion.

Proof. Following the hint, we will show that $(1, -1, 1), (1, 1, 1)$ is linearly independent. Suppose $a_1, a_2 \in \mathbb{F}$ satisfy

$$a_1(1, -1, 1) + a_2(1, 1, 1) = (0, 0, 0).$$

The left-hand side of the above equation becomes

$$(a_1 + a_2, -a_1 + a_2, a_1 + a_2) = (0, 0, 0),$$

which means we obtain a system of equations

$$\begin{aligned} a_1 + a_2 &= 0, \\ -a_1 + a_2 &= 0, \end{aligned}$$

from which system-solving gives $a_1 = 0, a_2 = 0$. Therefore, $(1, -1, 1), (1, 1, 1)$ is a linearly independent list. By 2.33 of Axler, we can extend this list to a basis of $\mathbb{R}^3$. Since $(1, 0), (0, 1)$ is the standard basis of $\mathbb{R}^3$, we can use 3.5 of Axler to assert that there exists a unique map satisfying

$$\begin{aligned} T(1, -1, 1) &= (1, 0) \\ T(1, 1, 1) &= (0, 1) \\ T(x_1, x_2, x_3) &= (y_1, y_2), \end{aligned}$$

where $(x_1, x_2, x_3) \in R^3$ is some value such that the resulting list $(1, -1, 1), (1, 1, 1), (x_1, x_2, x_3)$ is a basis of $\mathbb{R}^3$, and $(y_1, y_2) \in \mathbb{R}^2$ is the output from T of (x_1, x_2, x_3). All the three equations above came out of 3.5 of Axler, but only the first two are necessary to complete this proof. □

(8pts) b. Compute $T(-5, 1, -5)$.

Proof. Since T is linear, we can use additivity and homogeneity of T to get

$$\begin{aligned} T(-5, 1, -5) &= T((-3, 3, -3) + (-2, -2, -2)) \\ &= T(-3, 3, -3) + T(-2, -2, -2) \\ &= T(-3(1, -1, 1)) + T(-2(1, 1, 1)) \\ &= -3T(1, -1, 1) - 2T(1, 1, 1) \\ &= -3(1, 0) - 2(0, 1) \\ &= (-3, 0) + (0, -2) \\ &= (-3 + 0, 0 - 2) \\ &= (-3, -2), \end{aligned}$$

as desired. □

(20pts) 4. Suppose V and W are both finite-dimensional. Prove that there exists an injective map $T \in \mathcal{L}(V, W)$ if and only if $\dim V \leq \dim W$.

Proof. Forward direction: If there exists a injective linear map $T \in \mathcal{L}(V, W)$, then $\dim V \leq \dim W$. Suppose there exists a injective linear map $T \in \mathcal{L}(V, W)$, which means by 3.16 of Axler we have $\operatorname{null} T = \{0\}$. Since 3.19 of Axler says that $\operatorname{range} T$ is a subspace of W, by 2.38 of Axler, we have $\dim \operatorname{range} T \leq \dim W$. By the Fundamental Theorem of Linear Maps (3.22 of Axler), we have

$$\begin{aligned}\dim V &= \dim \operatorname{null} T + \dim \operatorname{range} T \\ &= \dim\{0\} + \dim \operatorname{range} T \\ &= 0 + \dim \operatorname{range} T \\ &= \dim \operatorname{range} T \\ &\leq \dim W,\end{aligned}$$

or $\dim W \leq \dim V$, as desired.

Backward direction: If $\dim V \leq \dim W$, then there exists an injective linear map $T \in \mathcal{L}(V, W)$. Suppose we have $\dim V \leq \dim W$. Since V and W are finite-dimensional, according to 2.32 of Axler, there exist a basis $v_1, \dots, v_n$ of V and a basis $w_1, \dots, w_m$ of W. For brevity in notation, let $m = \dim W$ and $n = \dim V$, which means $n \leq m$. Define $T : V \to W$ by

$$T(a_1 v_1 + \cdots + a_n v_n) = a_1 w_1 + \cdots + a_m w_m$$

for some $a_1, \dots, a_n, \dots, a_m \in \mathbb{F}$. Then T is linear and indeed defines a function, according to the proof for 3.5 in Axler. Now suppose we have $a_1 v_1 + \cdots + a_n v_n \in \operatorname{null} T$. Then we have $T(a_1 v_1 + \cdots + a_n v_n) = 0$, or

$$a_1 w_1 + \cdots + a_m w_m = 0.$$

Since $w_1, \dots, w_m$ is a basis of W, it is linearly independent, which means all the scalars are zero; that is, we have

$$a_1 = 0, \dots, a_m = 0.$$

Since $n \leq m$, we have in particular the first n of the m scalars are zero; that is, we have $a_1 = 0, \dots, a_m = 0$. So we have

$$a_1 v_1 + \cdots + a_n v_n = 0,$$

which means we have $\operatorname{null} T \subset \{0\}$. But 3.14 of Axler says that $\operatorname{null} T$ is a subspace in V, which means in particular that $\operatorname{null} T$ contains the additive identity, or $\{0\} \subset \operatorname{null} T$. Therefore, we have the set equality $\operatorname{null} T = \{0\}$. Finally, by 3.16 of Axler, T is injective. □

(20pts) 5. Suppose V and W are both finite-dimensional. Prove that there exists a surjective map $T \in \mathcal{L}(V, W)$ if and only if $\dim V \geq \dim W$.

Proof. Forward direction: If there exists a surjective linear map $T \in \mathcal{L}(V, W)$, then $\dim V \geq \dim W$. Suppose there exists a surjective map $T \in \mathcal{L}(V, W)$, which means we have $\operatorname{range} T = W$, and so $\dim \operatorname{range} T = \dim W$. Since T is a linear map, by 3.11 of Axler we have $T(0) = 0$. So we have $\{0\} \subset \operatorname{null} T$, and so, by 2.38 of Axler, we have $0 = \dim\{0\} \leq \dim \operatorname{null} T$. By the Fundamental Theorem of Linear Maps (3.22 of Axler), we have

$$\begin{aligned}\dim V &= \dim \operatorname{null} T + \dim \operatorname{range} T \\ &= \dim \operatorname{null} T + \dim W \\ &\geq \dim\{0\} + \dim W \\ &= 0 + \dim W \\ &= \dim W,\end{aligned}$$

as desired.

Backward direction: If $\dim V \geq \dim W$, then there exists a surjective map $T \in \mathcal{L}(V, W)$. Suppose we have $\dim V \leq \dim W$. Since V and W are finite-dimensional, according to 2.32 of Axler, there exist a basis of V and a basis of W. For brevity in notation, let $m = \dim W$ and $n = \dim V$, which means $n \geq m$. Define $T : V \to W$ by

$$T(a_1 v_1 + \cdots + a_n v_n) = a_1 w_1 + \cdots + a_m w_m$$

for some $a_1, \dots, a_m, \dots, a_n \in \mathbb{F}$. Then T is linear and indeed defines a function, according to the proof for 3.5 in Axler. Since $w_1, \dots, w_n$ is a basis of W, every vector in W is a linear combination of $w_1, \dots, w_n$ and can therefore be written $a_1 w_1 + \cdots + a_m w_m$. This implies that we have $\operatorname{range} T = W$, and so T is surjective. □