MATH 131: Linear Algebra I

University of California, Riverside

Group Examination 4

Time limit: 60 minutes

Score: _____ / 100

July 18, 2019

This group examination is open textbook, open lecture notes, open homework, and open classmates.

By writing my name and student ID number below, I agree to the following terms:

- I promise not to engage in any form of academic dishonesty. In particular, I will not use any resources other than what is listed above. I understand that any act of cheating may cause me to receive a failing grade in the course and further disciplinary action from the university.
- I will turn my cellular phone off and place it on the desk in front of me. If I do not have a cellular phone, I will notify the instructor before the start of any quiz or examination.
- If I need to use the restroom during any exam or quiz, then I must ask the instructor for permission. I cannot use the restroom for more than 15 minutes, more than once, or while another student is using the restroom. Also, I cannot take anything with me to the restroom. If I violate any of these policies, I understand that the instructor may dismiss me early and will only be graded for the work done.
- I will not open this booklet until the instructor tells the class to do so.

Name:______________________________ Student ID:______________________________

(20pts) 1. For this question, you will need to refer to the definitions in Chapter 3, Sections C-D, of your Axler textbook to find the answers.

(4pts) a. Write down the definitions of *matrix* and *matrix of a linear map*.

(6pts) b. Write down the definitions of *matrix addition*, *scalar multiplication of a matrix*, and *matrix multiplication*.

(4pts) c. Write down the definitions of an *invertible* linear map T and an *inverse* of T.

(4pts) d. Write down the definitions of an *isomorphism* and two vector spaces being *isomorphic*.

(2pts) e. Write down the definition of a *matrix of a vector*.

(20pts) 2. Suppose $T \in \mathcal{L}(\mathbb{F}^n, \mathbb{F}^m)$ has a matrix representation

$$\mathcal{M}(T) = \begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & \ddots & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{pmatrix}$$

with the respect to the standard bases of $\mathbb{F}^n$ and $\mathbb{F}^m$. Prove that we have

$$T(x_1, \ldots, x_n) = (A_{1,1}x_1 + \cdots + A_{1,n}x_n, \ldots, A_{m,1}x_1 + \cdots + A_{m,n}x_n)$$

for all $(x_1, \ldots, x_n) \in \mathbb{F}^n$.

Hint: Write $(x_1, \ldots, x_n) \in \mathbb{F}^n$ as a linear combination of standard basis vectors $(1, 0, \ldots, 0), (0, 1, 0, \ldots, 0), \ldots, (0, \ldots, 0, 1)$ of $\mathbb{F}^n$; namely, write

$$\begin{aligned} (x_1, \ldots, x_n) &= (x_1, 0, \ldots, 0) + (0, x_2, 0, \ldots, 0) + \cdots + (0, \ldots, 0, x_n) \\ &= x_1(1, 0, \ldots, 0) + x_2(0, 1, 0, \ldots, 0) + \cdots + x_n(0, \ldots, 0, 1). \end{aligned}$$

Then use 3.62 and 3.65 of Axler to compute $\mathcal{M}(T(x_1, \ldots, x_n))$, which you can use—along with the standard basis of $\mathbb{F}^m$—to find $T(x_1, \ldots, x_n)$.

(20pts) 3. Let V be a vector space over $\mathbb{F}$. suppose $v_1, \ldots, v_n$ is a basis of V. Prove that the map $T : V \to \mathbb{F}^{n,1}$ defined by

$$Tv = \mathcal{M}(v)$$

is an isomorphism of V onto $\mathbb{F}^{n,1}$; here $\mathcal{M}(v)$ is the matrix of $v \in V$ with respect to the basis $v_1, \ldots, v_n$.

(20pts) 4. Let V and W be vector spaces over $\mathbb{F}$ and suppose that $T : V \to W$ is an isomorphism. Prove that the map $\varphi : \mathcal{L}(V) \to \mathcal{L}(W)$ defined by

$$\varphi(S) = TST^{-1}$$

is an isomorphism.

Note: Observe that φ is a map between linear spaces $\mathcal{L}(V)$ and $\mathcal{L}(W)$, not between vector spaces V and W. Also, to clarify, we have $S \in \mathcal{L}(V)$ and $\varphi(S) = TST^{-1} \in \mathcal{L}(W)$, and $\mathcal{L}(V) = \mathcal{L}(V, V)$ and $\mathcal{L}(W) = \mathcal{L}(W, W)$.

(20pts) 5. Let V be a finite-dimensional vector space over $\mathbb{F}$, and suppose $T \in \mathcal{L}(V)$. Prove that T is a scalar multiple of the identity if and only if $ST = TS$ for all $S \in \mathcal{L}(V)$.