MATH 131: Linear Algebra I
University of California, Riverside
Group Examination 4 Solutions
July 18, 2019

(20pts) 1. For this question, you will need to refer to the definitions in Chapter 3, Sections C-D, of your Axler textbook to find the answers.

(4pts) a. Write down the definitions of *matrix* and *matrix of a linear map*.

Definition. From Definitions 3.30 and 3.32 of Axler:

- Let m, n be positive integers. An $m \times n$ matrix A is a rectangular array of elements of $\mathbb{F}$ with m rows and n columns:

$$A = \begin{pmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{m,1} & \cdots & a_{m,n} \end{pmatrix},$$

where the notation $a_{j,k}$ denotes the entry in row j, column k of A, for any $j = 1, \ldots, m$ and for any $k = 1, \ldots, n$.
- Suppose $T \in \mathcal{L}(V, W)$, and suppose $v_1, \ldots, v_n$ is a basis of V and $w_1, \ldots, w_n$ is a basis of W. The *matrix of a linear map* T with respect to these bases is the $m \times n$ matrix $\mathcal{M}(T)$ whose entries $a_{j,k}$ are defined by $Tv_k = a_{1,k}w_1 + \cdots + a_{m,k}w_m$.

(6pts) b. Write down the definitions of *matrix addition*, *scalar multiplication of a matrix*, and *matrix multiplication*.

Definition. From Definitions 3.35, 3.37, and 3.41 of Axler:

- *Matrix addition* is the sum of two matrices of the same size, which is obtained by adding corresponding entries in the matrices:

$$\begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & \ddots & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{pmatrix} + \begin{pmatrix} C_{1,1} & \cdots & C_{1,n} \\ \vdots & \ddots & \vdots \\ C_{m,1} & \cdots & C_{m,n} \end{pmatrix} = \begin{pmatrix} A_{1,1} + C_{1,1} & \cdots & A_{1,n} + C_{1,n} \\ \vdots & \ddots & \vdots \\ A_{m,1} + C_{m,1} & \cdots & A_{m,n} + C_{m,n} \end{pmatrix}.$$

In other words, we have $(A + C)_{j,k} = A_{j,k} + C_{j,k}$ for any $j = 1, \ldots, m$ and for any $k = 1, \ldots, n$.
- *Scalar multiplication of a matrix* is the product of a scalar and a matrix, which is obtained by multiplying each entry in the matrix by the scalar:

$$\lambda \begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & \ddots & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{pmatrix} = \begin{pmatrix} \lambda A_{1,1} & \cdots & \lambda A_{1,n} \\ \vdots & \ddots & \vdots \\ \lambda A_{m,1} & \cdots & \lambda A_{m,n} \end{pmatrix}.$$

In other words, we have $(\lambda A)_{j,k} = \lambda a_{j,k}$ for any $j = 1, \ldots, m$ and for any $k = 1, \ldots, n$.
- Let m, n, p be positive integers. Suppose A is an $m \times n$ matrix and C is an $n \times p$ matrix. Then AC is defined to be the $m \times p$ matrix whose entry in row j, column k is given by

$$(AC)_{j,k} = \sum_{r=1}^{n} A_{j,r} C_{r,k}.$$

(4pts) c. Write down the definitions of an *invertible* linear map T and an *inverse* of T.

Definition. From Definitions 3.53 of Axler:

- A map $T \in \mathcal{L}(V, W)$ is called *invertible* if there exists $S \in \mathcal{L}(W, V)$ such that $ST = I_V$ and $ST = I_W$, where I_V, I_W are respectively the identity maps on V, W.
- A map $S \in \mathcal{L}(W, V)$ satisfying $ST = I_V$ and $TS = I_W$ is called an *inverse* of T.

(4pts) d. Write down the definitions of an *isomorphism* and two vector spaces being *isomorphic*.

Definition. From Definitions 3.58 of Axler:

- An *isomorphism* is an invertible linear map.
- Two vector spaces are called *isomorphic* if there is an isomorphism from one vector space onto the other one.

(2pts) e. Write down the definition of a *matrix of a vector*.

Definition. From Definitions 3.62 of Axler:

- Suppose $v_1, \ldots, v_n$ is a basis of V. The *matrix of the vector* $v \in V$ with respect to this basis is the $n \times 1$ matrix

$$\mathcal{M}(v) = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix},$$

where $c_1, \ldots, c_n \in \mathbb{F}$ satisfy $v = c_1 v_1 + \cdots + c_n v_n$.

(20pts) 2. Suppose $T \in \mathcal{L}(\mathbb{F}^n, \mathbb{F}^m)$ has a matrix representation

$$\mathcal{M}(T) = \begin{pmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{m,1} & \cdots & a_{m,n} \end{pmatrix}$$

with the respect to the standard bases of $\mathbb{F}^n$ and $\mathbb{F}^m$. Prove that we have

$$T(x_1, \dots, x_n) = (a_{1,1}x_1 + \cdots + a_{1,n}x_n, \dots, a_{m,1}x_1 + \cdots + a_{m,n}x_n)$$

for all $(x_1, \dots, x_n) \in \mathbb{F}^n$.

Hint: Write $(x_1, \dots, x_n) \in \mathbb{F}^n$ as a linear combination of standard basis vectors $(1, 0, \dots, 0), (0, 1, 0, \dots, 0), \dots, (0, \dots, 0, 1)$ of $\mathbb{F}^n$; namely, write

$$\begin{aligned}(x_1, \dots, x_n) &= (x_1, 0, \dots, 0) + (0, x_2, 0, \dots, 0) + \cdots + (0, \dots, 0, x_n) \\ &= x_1(1, 0, \dots, 0) + x_2(0, 1, 0, \dots, 0) + \cdots + x_n(0, \dots, 0, 1).\end{aligned}$$

Then use 3.62 and 3.65 of Axler to compute $\mathcal{M}(T(x_1, \dots, x_n))$, which you can use—along with the standard basis of $\mathbb{F}^m$—to find $T(x_1, \dots, x_n)$.

Proof. According to the given hint, we write

$$\begin{aligned}(x_1, \dots, x_n) &= (x_1, 0, \dots, 0) + (0, x_2, 0, \dots, 0) + \cdots + (0, \dots, 0, x_n) \\ &= x_1(1, 0, \dots, 0) + x_2(0, 1, 0, \dots, 0) + \cdots + x_n(0, \dots, 0, 1).\end{aligned}$$

By 3.62 of Axler, we have

$$\mathcal{M}((x_1, \dots, x_n)) = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}.$$

By 3.65 of Axler, we have

$$\begin{aligned}\mathcal{M}(T(x_1, \dots, x_n)) &= \mathcal{M}(T)\mathcal{M}((x_1, \dots, x_n)) \\ &= \begin{pmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{m,1} & \cdots & a_{m,n} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \\ &= \begin{pmatrix} a_{1,1}x_1 + \cdots + a_{1,n}x_n \\ \vdots \\ a_{m,1}x_1 + \cdots + a_{m,n}x_n \end{pmatrix}.\end{aligned}$$

The standard basis of $\mathbb{F}^m$ is the list of the m vectors $(1, 0, \dots, 0), (0, 1, 0, \dots, 0), \dots, (0, 0, \dots, 0, 1)$. Using this basis and taking the entries of $\mathcal{M}(T(x_1, \dots, x_n))$, we obtain

$$\begin{aligned}T(x_1, \dots, x_n) &= (a_{1,1}x_1 + \cdots + a_{1,n}x_n)(1, 0, \dots, 0) + \cdots + (a_{m,1}x_1 + \cdots + a_{m,n}x_n)(0, \dots, 0, 1) \\ &= (a_{1,1}x_1 + \cdots + a_{1,n}x_n, \dots, a_{m,1}x_1 + \cdots + a_{m,n}x_n),\end{aligned}$$

as desired. □

(20pts) 3. Let V be a vector space over $\mathbb{F}$. Suppose $v_1, \dots, v_n$ is a basis of V. Prove that the map $T : V \longrightarrow \mathbb{F}^{n,1}$ defined by

$$Tv = \mathcal{M}(v)$$

is an isomorphism of V onto $\mathbb{F}^{n,1}$; here $\mathcal{M}(v)$ is the matrix of $v \in V$ with respect to the basis $v_1, \dots, v_n$.

Proof. To show that $T : V \longrightarrow \mathbb{F}^{n,1}$ is an isomorphism, we need to show that T is linear and invertible. First, we will show that T is linear. Since $v_1, \dots, v_n$ is a basis of V, there exist $a_1, \dots, a_n, b_1, \dots, b_n \in \mathbb{F}$ such that

$$u = a_1v_1 + \cdots + a_nv_n$$

and

$$v = b_1v_1 + \cdots + b_nv_n.$$

So, for all $u, v \in V$ and for all $\lambda \in \mathbb{F}$, we have

$$\begin{aligned} T(u+v) &= \mathcal{M}(u+v) \\ &= \mathcal{M}((a_1 v_1 + \cdots + a_n v_n) + (b_1 v_1 + \cdots + b_n v_n)) \\ &= \mathcal{M}((a_1 + b_1) v_1 + \cdots + (a_n + b_n) v_n) \\ &= \begin{pmatrix} a_1 + b_1 \\ \vdots \\ a_n + b_n \end{pmatrix} \\ &= \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} + \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \\ &= \mathcal{M}(a_1 v_1 + \cdots + a_n v_n) + \mathcal{M}(b_1 v_1 + \cdots + b_n v_n) \\ &= \mathcal{M}(u) + \mathcal{M}(v) \\ &= Tu + Tv, \end{aligned}$$

satisfying additivity, and

$$\begin{aligned} T(\lambda u) &= \mathcal{M}(\lambda u) \\ &= \mathcal{M}(\lambda(a_1 v_1 + \cdots + a_n v_n)) \\ &= \mathcal{M}((\lambda a_1) v_1 + \cdots + (\lambda a_n) v_n) \\ &= \begin{pmatrix} \lambda a_1 \\ \vdots \\ \lambda a_n \end{pmatrix} \\ &= \lambda \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \\ &= \lambda \mathcal{M}(a_1 v_1 + \cdots + a_n v_n) \\ &= \lambda \mathcal{M}(u) \\ &= \lambda T u, \end{aligned}$$

satisfying homogeneity. So T is linear. Next, we will show that T is invertible. According to 3.56 of Axler, this is equivalent to showing that T is injective and surjective. First, we will show that T is injective. Suppose $Tu = 0$. Since we have from earlier $u = a_1 v_1 + \cdots + a_n v_n$, we get

$$\begin{aligned} \mathcal{M}(a_1 v_1 + \cdots + a_n v_n) &= \mathcal{M}(u) \\ &= Tu \\ &= 0. \end{aligned}$$

We can write both sides of the above equation as matrices:

$$\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

So our scalars are $a_1 = 0, \ldots, a_n = 0$, which means

$$\begin{aligned} u &= a_1 v_1 + \cdots + a_n v_n \\ &= 0 v_1 + \cdots + 0 v_n \\ &= 0. \end{aligned}$$

So $\operatorname{null} T = \{0\}$, which means, by 3.16 of Axler, T is injective. Next, we will show that T is surjective. We have

$$\begin{aligned} Tu &= \mathcal{M}(u) \\ &= \mathcal{M}(a_1 v_1 + \cdots + a_n v_n) \\ &= \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}. \end{aligned}$$

Since $a_1, \ldots, a_n \in \mathbb{F}$ are arbitrary values, we conclude that T is also surjective. Therefore, T is both injective and surjective, which means T is invertible. Moreover, T is both linear and invertible, which means T is an isomorphism. □

(20pts) 4. Let V and W be vector spaces over $\mathbb{F}$ and suppose that $T : V \to W$ is an isomorphism. Prove that the map $\varphi : \mathcal{L}(V) \to \mathcal{L}(W)$ defined by

$$\varphi(S) = TST^{-1}$$

is an isomorphism.

Note: Observe that φ is a map between linear spaces $\mathcal{L}(V)$ and $\mathcal{L}(W)$, not between vector spaces V and W. Also, to clarify, we have $S \in \mathcal{L}(V)$ and $\varphi(S) = TST^{-1} \in \mathcal{L}(W)$, and $\mathcal{L}(V) = \mathcal{L}(V, V)$ and $\mathcal{L}(W) = \mathcal{L}(W, W)$.

Proof. To show that $\varphi : \mathcal{L}(V) \to \mathcal{L}(W)$ is an isomorphism, we need to show that φ is linear and invertible. First, we will show that φ is linear.

- Additivity: For all $R, S \in \mathcal{L}(V)$, we apply the left and right distributive properties of $\mathcal{L}(V)$ to obtain

$$\begin{aligned}\varphi(R + S) &= T(R + S)T^{-1} \\ &= (TR + TS)T^{-1} \\ &= (TR)T^{-1} + (TS)T^{-1} \\ &= TRT^{-1} + TST^{-1} \\ &= \varphi(R) + \varphi(S).\end{aligned}$$

- Homogeneity: For all $\lambda \in \mathbb{F}$ and for all $R \in \mathcal{L}(V)$, we have

$$\begin{aligned}\varphi(\lambda S) &= T(\lambda S)T^{-1} \\ &= T(\lambda(ST^{-1})) \\ &= \lambda T(ST^{-1}) \\ &= \lambda(TST^{-1}) \\ &= \lambda\varphi(S).\end{aligned}$$

Since the two properties of a linear map are satisfied, this map is linear. Finally, we will show that φ is injective. Suppose we have $S \in \operatorname{null} \varphi$, which means we have $\varphi(S) = 0$. So we get

$$\begin{aligned}S &= I_V S I_V \\ &= (T^{-1}T)S(T^{-1}T) \\ &= T^{-1}(TST^{-1})T \\ &= T^{-1}\varphi(S)T \\ &= T^{-1}0T \\ &= T^{-1}(0(T)) \\ &= T^{-1}(0) \\ &= 0,\end{aligned}$$

where I_V is the identity map on V. Therefore, we conclude $\operatorname{null} \varphi \subset \{0\}$. At the same time, 3.14 of Axler states that $\operatorname{null} \varphi$ is a subspace of $\mathcal{L}(V)$, which means in particular that we have $\{0\} \subset \operatorname{null} \varphi$. So we conclude the set equality $\operatorname{null} \varphi = \{0\}$. Finally, by 3.16 of Axler, we conclude that φ is injective. Next, we will show that φ is surjective. Suppose we have $U \in \mathcal{L}(W)$. Then we have $T^{-1}UT \in \mathcal{L}(V)$ and

$$\begin{aligned}\varphi(T^{-1}UT) &= T(T^{-1}UT)T^{-1} \\ &= (TT^{-1})U(TT^{-1}) \\ &= I_W U I_W \\ &= U,\end{aligned}$$

where I_W is the identity map on W. So we get $U \in \operatorname{range} \varphi$, and so we have the set containment $\mathcal{L}(W) \subset \operatorname{range} \varphi$. But 3.19 of Axler says that $\operatorname{range} \varphi$ is a subsapce of $\mathcal{L}(W)$. So we conclude the set equality $\operatorname{range} \varphi = \mathcal{L}(W)$, which means φ is surjective. So we have established that φ is both injective and surjective, which implies, by 3.56 of Axler, that φ is invertible. Therefore, φ is both linear and invertible, which means it is an isomorphism. □

(20pts) 5. Let V be a finite-dimensional vector space over $\mathbb{F}$, and suppose $T \in \mathcal{L}(V)$. Prove that T is a scalar multiple of the identity if and only if $ST = TS$ for all $S \in \mathcal{L}(V)$.

Proof. Forward direction: If T is a sclar multiple of the identity, then $ST = TS$ for all $S \in \mathcal{L}(V)$. Suppose T is the scalar multiple of the identity map on V. Then there exists $\lambda \in \mathbb{F}$ such that we have $T = \lambda I_V$, where I_V is the identity map on V.

Therefore, for all $S \in \mathcal{L}(V)$, we get

$$\begin{aligned} ST &= S(\lambda I_V) \\ &= \lambda S I_V \\ &= \lambda S \\ &= (\lambda I_V)S \\ &= TS, \end{aligned}$$

as desired.

Backward direction: If $ST = TS$ for all $S \in \mathcal{L}(V)$, then T is a sclar multiple of the identity. Suppose that we have $ST = TS$ for all $S \in \mathcal{L}(V)$. First, we will show that, for all $v \in V$, the list v, Tv is linearly dependent. Suppose instead by contradiction that v, Tv is linearly independent. Then, according to 2.33 of Axler, we can extend v, Tv to a basis $v, Tv, u_1, \ldots, u_n$ of V. (This means the dimension of V is $\dim V = n + 2$, but that is not really important in this proof.) So every vector in V can be written in the form $av + bTv + c_1u_1 + \cdots + c_nu_n$ for some $a, b, c_1, \ldots, c_n \in \mathbb{F}$. This means that we can define $S \in \mathcal{L}(V)$ by

$$S(av + bTv + c_1u_1 + \cdots + c_nu_n) = bv,$$

which satisfies in particular $S(Tv) = v$ and $Sv = 0$. Therefore, since $ST = TS$, we obtain

$$\begin{aligned} v &= S(Tv) \\ &= (ST)v \\ &= (TS)v \\ &= T(Sv) \\ &= T(0) \\ &= 0, \end{aligned}$$

using 3.11 of Axler to justify the last equality above. So we can choose nonzero scalars such as $a_1 = 1, a_2 = 1 \in \mathbb{F}$ to satisfy

$$\begin{aligned} a_1v + a_2Tv &= (1)(0) + (1)T(0) \\ &= (1)(0) + (1)(0) \\ &= 0 + 0 \\ &= 0, \end{aligned}$$

meaning that the list v, Tv is linearly dependent. But this contradicts our assumption at the beginning that v, Tv is linearly independent. Therefore, the list v, Tv must be linearly dependent. By the Linear Dependence Lemma (2.21 of Axler), we have $Tv \in \operatorname{span}(v)$. In other words, for all nonzero $v \in V$, there exists $\lambda_v \in \mathbb{F}$ (the subscript notation signifies that the scalar λ_v depends on our choice of some vector v) such that $Tv = \lambda_v v$, which means $T = \lambda_v I_V$, where once again I_V is the identity map on V. At this stage, we have almost completed our proof. To show that T is a scalar multiple of the identity, we need to establish $T = \lambda I_V$, where $\lambda \in \mathbb{F}$ does not depend on v. In other words, it is not enough to stop at $Tv = \lambda_v v$; we need to show that λ_v is actually constant in v, at which point would allow us to write $\lambda_v = \lambda$. Let $w \in V$ be another arbitrary vector. Then v, w form a list that is either linearly independent or linearly dependent. Consider $\lambda_v, \lambda_w, \lambda_{v+w} \in \mathbb{F}$, the scalars that depend on $v, w, v + w$, respectively. In the first case, assume that v, w is linearly independent. Applying $Tv = \lambda_v v$, we obtain

$$\begin{aligned} (\lambda_{v+w} - \lambda_v)v + (\lambda_{v+w} - \lambda_w)w &= \lambda_{v+w}v - \lambda_v v + \lambda_{v+w}w - \lambda_w w \\ &= \lambda_{v+w}(v + w) - \lambda_v v - \lambda_w w \\ &= T(v + w) - \lambda_v v - \lambda_w w \\ &= Tv + Tw - \lambda_v v - \lambda_w w \\ &= \lambda_v v + \lambda_w w - \lambda_v v - \lambda_w w \\ &= 0. \end{aligned}$$

Since v, w is linearly independent, all scalars are zero; that is, we have

$$\lambda_{v+w} - \lambda_v = 0, \lambda_{v+w} - \lambda_w = 0,$$

or $\lambda_v = \lambda_{v+w} = \lambda_w$. Any function that outputs the same value such as $\lambda_v = \lambda_w$ for all input values such as $v, w \in V$ must be a constant function; in other words, we conclude that λ_v is constant, or $\lambda_v = \lambda$. Therefore, we conclude $T = \lambda_v I_V = \lambda I_V$, which means that T is a scalar multiple of the identity. □