MATH 131: Linear Algebra I

University of California, Riverside

Group Examination 5

Time limit: 60 minutes

Score: _____ / 100

July 25, 2019

This group examination is open textbook, open lecture notes, open homework, and open classmates.

By writing my name and student ID number below, I agree to the following terms:

- I promise not to engage in any form of academic dishonesty. In particular, I will not use any resources other than what is listed above. I understand that any act of cheating may cause me to receive a failing grade in the course and further disciplinary action from the university.
- I will turn my cellular phone off and place it on the desk in front of me. If I do not have a cellular phone, I will notify the instructor before the start of any quiz or examination.
- If I need to use the restroom during any exam or quiz, then I must ask the instructor for permission. I cannot use the restroom for more than 15 minutes, more than once, or while another student is using the restroom. Also, I cannot take anything with me to the restroom. If I violate any of these policies, I understand that the instructor may dismiss me early and will only be graded for the work done.
- I will not open this booklet until the instructor tells the class to do so.

Name:____________________________ Student ID:____________________________

(20pts) 1. For this question, you will need to refer to the definitions in Chapter 3, Sections E-F, of your Axler textbook to find the answers.

(4pts) a. Write down the definition of $v + U$ and the *quotient space* V/U.

(4pts) b. Write down the definitions of *addition* and *scalar multiplication* on V/U.

(2pts) c. Write down the definition of a *quotient map*.

(4pts) d. Write down the definitions of a *linear functional* and a *dual space*.

(4pts) e. Write down the definitions of a *dual basis* and a *dual map*.

(2pts) f. Write down the definition of the *transpose* of a matrix A.

(20pts) 2. Suppose $T \in \mathcal{L}(V, W)$ and U is a subspace of V. Let $\pi : V \to V/U$ be the quotient map. Prove that there exists $S \in \mathcal{L}(V/U, W)$ such that $T = S \circ \pi$ if and only if $U \subset \operatorname{null} T$.

(20pts) 3. Suppose U is a subspace of V. Define $\Gamma : \mathcal{L}(V/U, W) \to \mathcal{L}(V, W)$ by

$$\Gamma(S) = S \circ \pi.$$

(8pts) a. Show that Γ is a linear map.

(6pts) b. Show that Γ is injective.

(6pts) c. Show that $\operatorname{range} \Gamma = \{T \in \mathcal{L}(V, W) : Tu = 0 \text{ for all } u \in U\}$.

(20pts) 4. We will compute the dual basis of some basis in $\mathbb{R}^3$.

(12pts) a. Show that the list $(1, 0, -1), (1, 1, 1), (2, 2, 0)$ is a basis of $\mathbb{R}^3$.

(8pts) b. What is the dual basis of the basis in part (a)?

(20pts) 5. Suppose V is a finite-dimensional vector space and $T \in \mathcal{L}(V, W)$. We will construct a different proof of the Fundamental Theorem of Linear Maps (3.22 of Axler) using quotient spaces and isomorphisms.

Note: Avoid using any theorem from the Axler textbook if its corresponding proof depends on the Fundamental Theorem of Linear Maps. Because we are completing a different proof of the Fundamental Theorem of Linear Maps, any attempt to cite those results here creates the logical fallacy of circular reasoning. However, you may use any result from Axler whose proofs do not depend on the Fundamental Theorem of Linear Maps.

(15pts) a. We recall from 3.91 of Axler that $V/(\operatorname{null} T)$ is isomorphic to $\operatorname{range} T$ and that $\tilde{T} : V/(\operatorname{null} T) \to W$ defined by

$$\tilde{T}(v + \operatorname{null} T) = Tv$$

is an isomorphism. Use the isomorphism to prove that $v_1 + \operatorname{null} T, \ldots, v_n + \operatorname{null} T$ is a basis of $V/(\operatorname{null} T)$ if and only if $Tv_1, \ldots, Tv_n$ is a basis of $\operatorname{range} T$. Without using 3.59 of Axler, conclude that we have

$$\dim(V/(\operatorname{null} T)) = \dim \operatorname{range} T.$$

(5pts) b. Use part (a) of this question and Exercise 3.E.13 of Axler to show that $\operatorname{range} T$ is finite-dimensional and that we have

$$\dim V = \dim \operatorname{null} T + \dim \operatorname{range} T.$$

Remark: This is precisely the assertion of the Fundamental Theorem of Linear Maps (3.22 of Axler).