MATH 131: Linear Algebra I
University of California, Riverside
Group Examination 5 Solutions
July 25, 2019

(20pts) 1. For this question, you will need to refer to the definitions in Chapter 3, Sections E-F, of your Axler textbook to find the answers.

(4pts) a. Write down the definition of $v + U$ and the *quotient space* V/U.

Definition. From Definitions 3.79 and 3.83 of Axler: Suppose U is a subspace of V.

- Suppose we have $v \in V$. Then $v + U$ is the subset of V defined by $v + U = \{v + u : u \in U\}$.
- The *quotient space* V/U is the set of all affine subsets of V parallel to U; that is, we have $V/U = \{v + U : v \in V\}$.

(4pts) b. Write down the definitions of *addition* and *scalar multiplication* on V/U.

Definition. From Definition 3.86 of Axler: Suppose U is a subspace of V.

- *Addition* is defined on V/U by $(v + U) + (w + U) = (v + w) + U$ for all $v, w \in V$.
- *Scalar multiplication* is defined on V/U by $\lambda(v + U) = (\lambda v) + U$ for all $\lambda \in \mathbb{F}$ and for all $v \in V$.

(2pts) c. Write down the definition of a *quotient map*.

Definition. From Definition 3.88 of Axler: Suppose U is a subspace of V.

- The *quotient map* $\pi : V \longrightarrow V/U$ is a linear map defined by $\pi(v) = v + U$ for all $v \in V$.

(4pts) d. Write down the definitions of a *linear functional* and a *dual space*.

Definition. From Definitions 3.92 and 3.94 of Axler:

- A *linear functional* on V is a linear map from V to $\mathcal{F}$; that is, it is an element of $\mathcal{L}(V, \mathbb{F})$.
- The *dual space* of V, denoted V', is the vector space of all linear funtionals on V; that is, we haave $V' = \mathcal{L}(V, \mathbb{F})$.

(4pts) e. Write down the definitions of a *dual basis* and a *dual map*.

Definition. From Definition 3.96 and 3.99 of Axler:

- If $v_1, \ldots, v_n$ is a basis of V, then the *dual basis* of $v_1, \ldots, v_n$ is the list $\varphi_1, \ldots, \varphi_n$ of elements of V', where each φ_j is the linear functional on V such that we have

$$\varphi_j(v_k) = \begin{cases} 1 & \text{if } k = j, \\ 0 & \text{if } k \neq j. \end{cases}$$

- If $T \in \mathcal{L}(V, W)$, then the *dual map* of T is the linear map $T' \in \mathcal{L}(W', V')$ deifned by $T'(\varphi) = \varphi \circ T$ for all $\varphi \in W'$.

(2pts) f. Write down the definition of the *transpose* of a matrix A.

Definition. From Definition 3.111 of Axler:

- The *transpose* of a matrix A, denoted A^t, is the matrix obatined from A by interchanging the rowns and columns. More specifically, if A is an $m \times n$ matrix, then A^t is the $n \times m$ matrix whose entries are given by the equation $(A^t)_{k,j} = A_{j,k}$.

(20pts) 2. Suppose $T \in \mathcal{L}(V, W)$ and U is a subspace of V. Let $\pi : V \longrightarrow V/U$ be the quotient map. Prove that there exists $S \in \mathcal{L}(V/U, W)$ such that $T = S \circ \pi$ if and only if $U \subset \operatorname{null} T$.

Proof. Forward direction: If there exists $S \in \mathcal{L}(V/U, W)$ such that $T = S \circ \pi$, then $U \subset \operatorname{null} T$. Suppose there exists $S \in \mathcal{L}(V/U), W)$ such that $T = S \circ \pi$. Let $u \in U$ be arbitrary. We have $v - 0 = v \in U$, and so, by 3.85—(a) implies (b)—of Axler, we have $v + U = 0 + U$. So, using 3.88 of Axler, for all $u \in U$, we have

$$\begin{aligned} Tu &= (S \circ \pi)u \\ &= S(\pi(u)) \\ &= S(u + U) \\ &= S(0 + U) \\ &= 0, \end{aligned}$$

where we also used 3.11 of Axler in the last equality above. So we have $u \in \operatorname{null} T$, and so we conclude $U \subset \operatorname{null} T$.

Backward direction: If $U \subset \operatorname{null} T$, then there exists $S \in \mathcal{L}(V/U, W)$ such that $T = S \circ \pi$. Suppose that we have $U \subset \operatorname{null} T$. Let $v \in V$ be arbitrary, and define $S : V/U \longrightarrow W$ by

$$S(v + U) = Tv.$$

Consider another vector $\hat{v} \in V$ that satisfies $v + U = \hat{v} + U$. Since we assumed $U \subset \operatorname{null} T$, we have $v - \hat{v} \in \operatorname{null} T$, which means we have $T(v - \hat{v}) = 0$. So we get

$$\begin{aligned} S(v+U) &= Tv \\ &= T((v-\hat{v})+\hat{v}) \\ &= T(v-\hat{v}) + T\hat{v} \\ &= 0 + T\hat{v} \\ &= T\hat{v} \\ &= S(\hat{v}+U), \end{aligned}$$

which means S indeed defines a function. Next, we need to show that S is linear, given already that T is linear. For all $\lambda \in \mathbb{F}$ and for all $v, w \in V$, we have

$$\begin{aligned} S((v+U)+(w+U)) &= S((v+w)+U) \\ &= T(v+w) \\ &= Tv + Tw \\ &= S(v+U) + S(w+U), \end{aligned}$$

satisfying additivity, and

$$\begin{aligned} S(\lambda(v+U)) &= S(\lambda v + U) \\ &= T(\lambda v) \\ &= \lambda T v \\ &= \lambda S(v+U), \end{aligned}$$

satisfying homogeneity. So S is linear. Finally, for all $v \in V$, we have

$$\begin{aligned} (S \circ \pi)v &= S(\pi(v)) \\ &= S(v+V) \\ &= Tv, \end{aligned}$$

from which we conclude $T = S \circ \pi$. □

(20pts) 3. Suppose U is a subspace of V. Define $\Gamma : \mathcal{L}(V/U, W) \to \mathcal{L}(V, W)$ by

$$\Gamma(S) = S \circ \pi.$$

(8pts) a. Show that Γ is a linear map.

Proof. For all $\lambda \in \mathbb{F}$ and for all $S, T \in \mathcal{L}(V/U, W)$, we have

$$\begin{aligned} \Gamma(S+T) &= (S+T) \circ \pi \\ &= S \circ \pi + T \circ \pi \\ &= \Gamma(S) + \Gamma(T), \end{aligned}$$

satisfying additivity, and

$$\begin{aligned} \Gamma(\lambda S) &= (\lambda S) \circ \pi \\ &= \lambda S \circ \pi \\ &= \lambda \Gamma(S), \end{aligned}$$

satisfying homogeneity. So Γ is linear. □

(6pts) b. Show that Γ is injective.

Proof. Suppose we have $S \in \operatorname{null}\Gamma$, which means $\Gamma(S) = 0$. Then we have $S \circ \pi = \Gamma(S) = 0$, and so for all $v \in V$ we have $(S \circ \pi)v = 0$. Therefore,

$$\begin{aligned} 0 &= (S \circ \pi)v \\ &= S(\pi(v)) \\ &= S(v+U). \end{aligned}$$

Since $v \in V$ is arbitrary, we must have $S = 0$, and so $\operatorname{null}\Gamma \subset \{0\}$. But 3.14 of Axler says that $\operatorname{null}\Gamma$ is a subspace in V, which means in particular that $\operatorname{null}\Gamma$ contains the additive identity, or $\{0\} \subset \operatorname{null}\Gamma$. Therefore, we have the set equality $\operatorname{null}\Gamma = \{0\}$. Finally, by 3.16 of Axler, Γ is injective. □

(6pts) c. Show that range $\Gamma = \{T \in \mathcal{L}(V, W) : Tu = 0 \text{ for all } u \in U\}$.

Proof. By Exercise 3.E.18 of Axler (or Question 2 of this examination), there exists $S \in \mathcal{L}(V/U, W)$ satisfying $T = S \circ \pi$ if and only if we have $U \subset \operatorname{null} T$. Therefore, we have

$$\begin{aligned} \operatorname{range} \Gamma &= \{\Gamma(S) \in \mathcal{L}(V, W) : S \in \mathcal{L}(V/U, W)\} \\ &= \{S \circ \pi \in \mathcal{L}(V, W) : S \in \mathcal{L}(V/U, W)\} \\ &= \{T \in \mathcal{L}(V, W) : T = S \circ \pi, S \in \mathcal{L}(V/U, W)\} \\ &= \{T \in \mathcal{L}(V, W) : U \subset \operatorname{null} T\} \\ &= \{T \in \mathcal{L}(V, W) : Tu = 0 \text{ for all } u \in U\}, \end{aligned}$$

as desired. □

(20pts) 4. We will compute the dual basis of some basis in $\mathbb{R}^3$.

(12pts) a. Show that the list $(1, 0, -1), (1, 1, 1), (2, 2, 0)$ is a basis of $\mathbb{R}^3$.

Proof. First, we need to show that $(1, 0, -1), (1, 1, 1), (2, 2, 0)$ is a linearly independent set that spans $\mathbb{R}^3$. To do this, suppose $a_1, a_2, a_3 \in \mathbb{F}$ satisfy

$$a_1(1, 0, -1) + a_2(1, 1, 1) + a_3(2, 2, 0) = (0, 0, 0).$$

Then we have

$$\begin{aligned} (0, 0, 0) &= a_1(1, 0, -1) + a_2(1, 1, 1) + a_3(2, 2, 0) \\ &= (a_1, 0, -a_1) + (a_2, a_2, a_2) + (2a_3, 2a_3, 0) \\ &= (a_1 + a_2 + 2a_3, a_2 + 2a_3, -a_1 + a_2). \end{aligned}$$

Equating the coordinates gives us the system of equations

$$\begin{aligned} a_1 + a_2 + 2a_3 &= 0, \\ a_2 + 2a_3 &= 0, \\ -a_1 + a_2 &= 0, \end{aligned}$$

from which system-solving gives $a_1 = 0, a_2 = 0, a_3 = 0$. Therefore, $(1, 0, -1), (1, 1, 1), (2, 2, 0)$ is a linearly independent list. Furthermore, since $(1, 0, -1), (1, 1, 1), (2, 2, 0)$ has length 3 and we have $\dim \mathbb{R}^3 = 3$, it is of the right length, which means, by 2.39 of Axler, this list is a basis of $\mathbb{R}^3$. □

Alternate proof. First, we will prove that the list $(1, 0, -1), (1, 1, 1), (2, 2, 0)$ spans $\mathbb{R}^3$. This means we need to show that, for all vectors $(x_1, x_2, x_3) \in \mathbb{R}^3$, there exist $a_1, a_3, a_3 \in \mathbb{F}$ such that

$$(x_1, x_2, x_3) = a_1(1, 0, -1) + a_2(1, 1, 1) + a_3(2, 2, 0).$$

With that said, we have

$$\begin{aligned} (x_1, x_2, x_3) &= a_1(1, 0, -1) + a_2(1, 1, 1) + a_3(2, 2, 0) \\ &= (a_1, 0, -a_1) + (a_2, a_2, a_2) + (2a_3, 2a_3, 0) \\ &= (a_1 + a_2 + 2a_3, a_2 + 2a_3, -a_1 + a_2). \end{aligned}$$

Equating the coordinates gives us the system of equations

$$\begin{aligned} a_1 + a_2 + 2a_3 &= x_1, \\ a_2 + 2a_3 &= x_2, \\ -a_1 + a_2 &= x_3, \end{aligned}$$

from which system-solving gives

$$\begin{aligned} a_1 &= x_1 - x_2, \\ a_2 &= -x_1 + x_2 + x_3, \\ a_3 &= \frac{1}{3}x_1 - \frac{1}{2}x_3. \end{aligned}$$

So we found $a_1, a_2, a_3 \in \mathbb{F}$ that depend on the coordinates of the $(x_1, x_2, x_3) \in \mathbb{R}^3$, which means $(1, 0, -1), (1, 1, 1), (2, 2, 0)$ spans $\mathbb{R}^3$. Next, we need to prove the list $(1, 0, -1), (1, 1, 1), (2, 2, 0)$ is linearly independent. To do this, suppose $a_1, a_2, a_3 \in \mathbb{F}$ satisfy

$$a_1(1, 0, -1) + a_2(1, 1, 1) + a_3(2, 2, 0) = (0, 0, 0).$$

The above equation is only

$$(x_1, x_2, x_3) = a_1(1, 0, -1) + a_2(1, 1, 1) + a_3(2, 2, 0)$$

with $x_1 = 0, x_2 = 0, x_3 = 0$. Substituting $x_1 = 0, x_2 = 0, x_3 = 0$ into our expressions of the scalars a_1, a_2, a_3, we get

$$\begin{aligned} a_1 &= 0 - 0 = 0 \\ a_2 &= -0 + 0 + 0 = 0, \\ a_3 &= \frac{1}{3}(0) - \frac{1}{2}(0) = 0. \end{aligned}$$

So the list $(1, 0, -1), (1, 1, 1), (2, 2, 0)$ is linearly independent. Therefore, $(1, 0, -1), (1, 1, 1), (2, 2, 0)$ is a basis of $\mathbb{R}^3$. □

(8pts) b. What is the dual basis of the basis in part (a)?

Proof. Define the linear functional $\varphi : \mathbb{F}^3 \to \mathbb{F}$ by

$$\varphi_j(x_1, x_2, x_3) = c_1x_1 + c_2x_2 + c_3x_3$$

for all $j = 1, 2, 3$. Let $v_1 = (1, 0, 1), v_2 = (1, 1, 1), v_3 = (2, 2, 0) \in \mathbb{F}^3$. First, we have

$$\begin{aligned} 1 &= \varphi_1(v_1) = \varphi_1(1, 0, -1) = c_1(1) + c_2(0) + c_3(-1) = c_1 - c_3, \\ 0 &= \varphi_1(v_2) = \varphi_1(1, 1, 1) = c_1(1) + c_2(1) + c_3(1) = c_1 + c_2 + c_3, \\ 0 &= \varphi_1(v_3) = \varphi_1(2, 2, 0) = c_1(2) + c_2(2) + c_3(0) = 2c_1 + 2c_2, \end{aligned}$$

upon which system-solving gives $c_1 = 1, c_2 = -1, c_3 = 0$, and so $\varphi_1(x_1, x_2, x_3) = x_1 - x_2$. Next, we have

$$\begin{aligned} 0 &= \varphi_2(v_1) = \varphi_2(1, 0, -1) = c_1(1) + c_2(0) + c_3(-1) = c_1 - c_3, \\ 1 &= \varphi_2(v_2) = \varphi_2(1, 1, 1) = c_1(1) + c_2(1) + c_3(1) = c_1 + c_2 + c_3, \\ 0 &= \varphi_2(v_3) = \varphi_2(2, 2, 0) = c_1(2) + c_2(2) + c_3(0) = 2c_1 + 2c_2, \end{aligned}$$

upon which system-solving gives $c_1 = 1, c_2 = -1, c_3 = -\frac{1}{2}$, and so $\varphi_2(x_1, x_2, x_3) = x_1 - x_2 + x_3$. Finally, we have

$$\begin{aligned} 0 &= \varphi_3(v_1) = \varphi_3(1, 0, -1) = c_1(1) + c_2(0) + c_3(-1) = c_1 - c_3, \\ 0 &= \varphi_3(v_2) = \varphi_3(1, 1, 1) = c_1(1) + c_2(1) + c_3(1) = c_1 + c_2 + c_3, \\ 1 &= \varphi_3(v_3) = \varphi_3(2, 2, 0) = c_1(2) + c_2(2) + c_3(0) = 2c_1 + 2c_2, \end{aligned}$$

upon which system-solving gives $c_1 = -\frac{1}{2}, c_2 = 1, c_3 = -\frac{1}{2}$, and so $\varphi_3(x_1, x_2, x_3) = -\frac{1}{2}x_1 + x_2 - \frac{1}{2}x_3$. So $\varphi_1, \varphi_2, \varphi_3$ is the dual basis of the basis $(1, 0, -1), (1, 1, 1), (2, 2, 0)$. □

(20pts) 5. Suppose V is a finite-dimensional vector space and $T \in \mathcal{L}(V, W)$. We will construct a different proof of the Fundamental Theorem of Linear Maps (3.22 of Axler) using quotient spaces and isomorphisms.

Note: Avoid using any theorem from the Axler textbook if its corresponding proof depends on the Fundamental Theorem of Linear Maps. Because we are completing a different proof of the Fundamental Theorem of Linear Maps, any attempt to cite those results here creates the logical fallacy of circular reasoning. However, you may use any result from Axler whose proofs do not depend on the Fundamental Theorem of Linear Maps.

(15pts) a. We recall from 3.91 of Axler that $V/(\operatorname{null} T)$ is isomorphic to $\operatorname{range} T$ and that $\tilde{T} : V/(\operatorname{null} T) \to W$ defined by

$$\tilde{T}(v + \operatorname{null} T) = Tv$$

is an isomorphism. Use this isomorphism to prove that $v_1 + \operatorname{null} T, \ldots, v_n + \operatorname{null} T$ is a basis of $V/(\operatorname{null} T)$ if and only if $Tv_1, \ldots, Tv_n$ is a basis of $\operatorname{range} T$. Conclude that the dimensions of $V/(\operatorname{null} T)$ and $\operatorname{range} T$ are equal; in other words, establish

$$\dim(V/(\operatorname{null} T)) = \dim \operatorname{range} T.$$

Proof. Forward direction: If $v_1 + \operatorname{null} T, \ldots, v_n + \operatorname{null} T$ is a basis of $V/(\operatorname{null} T)$, then $Tv_1, \ldots, Tv_n$ is a basis of $\operatorname{range} T$. First, we will show that $Tv_1, \ldots, Tv_n$ is linearly independent in $\operatorname{range} T$. Suppose $a_1Tv_1 + \cdots + a_nTv_n = 0$. Then, since $\tilde{T}$ is an isomorphism, it is linear, and so we have

$$\begin{aligned} 0 &= a_1Tv_1 + \cdots + a_nTv_n \\ &= a_1\tilde{T}(v_1 + \operatorname{null} T) + \cdots + a_n\tilde{T}(v_n + \operatorname{null} T) \\ &= \tilde{T}(a_1(v_1 + \operatorname{null} T) + \cdots + a_n(v_n + \operatorname{null} T)). \end{aligned}$$

Since $\tilde{T}$ is an isomorphism, it is injective. By 3.16 of Axler, we have $\operatorname{null} \tilde{T} = \{0 + \operatorname{null} T\}$, and so we must have

$$a_1(v_1 + \operatorname{null} T) + \cdots + a_n(v_n + \operatorname{null} T) = 0 + \operatorname{null} T.$$

Since $v_1 + \operatorname{null} T, \dots, v_n + \operatorname{null} T$ is a basis of $V/(\operatorname{null} T)$, we must have

$$a_1 = 0, \dots, a_n = 0.$$

So $Tv_1, \dots, Tv_n$ is linearly independent in range T. Next, we will show that $Tv_1, \dots, Tv_n$ spans range T. Suppose we have $w = \operatorname{range} T$. Then $w = \tilde{T}v$ for some $v \in V$. Since $v_1 + \operatorname{null} T, \dots, v_n + \operatorname{null} T$ is a basis of $V/(\operatorname{null} T)$, it spans $V/(\operatorname{null} T)$, meaning that every vector in $V/(\operatorname{null} T)$ can be written uniquely as

$$v + \operatorname{null} T = a_1(v_1 + \operatorname{null} T) + \cdots + a_n(v_n + \operatorname{null} T)$$

for some $a_1, \dots, a_n \in \mathbb{F}$. Since T is linear, we have

$$\begin{aligned} w &= Tv \\ &= \tilde{T}(v + \operatorname{null} T) \\ &= \tilde{T}(a_1(v_1 + \operatorname{null} T) + \cdots + a_n(v_n + \operatorname{null} T)) \\ &= \tilde{T}(a_1(v_1 + \operatorname{null} T)) + \cdots + \tilde{T}(a_n(v_n + \operatorname{null} T)) \\ &= a_1\tilde{T}(v_1 + \operatorname{null} T) + \cdots + a_n\tilde{T}(v_n + \operatorname{null} T) \\ &= a_1Tv_1 + \cdots + a_nTv_n. \end{aligned}$$

So every $w \in \operatorname{range} T$ is a linear combination of $Tv_1, \dots, Tv_n$. Therefore, we conclude that $Tv_1, \dots, Tv_n$ is a basis of range T.

Backward direction: If $Tv_1, \dots, Tv_n$ is a basis of range T, then $v_1 + \operatorname{null} T, \dots, v_m + \operatorname{null} T$ is a basis of $V/(\operatorname{null} T)$. First, we will show that $v_1 + \operatorname{null} T, \dots, v_m + \operatorname{null} T$ is linearly independent in $V/(\operatorname{null} T)$. Suppose $b_1, \dots, b_n \in \mathbb{F}$ satisfy

$$b_1(v_1 + \operatorname{null} T) + \cdots + b_n(v_n + \operatorname{null} T) = 0 + \operatorname{null} T.$$

Then, since $\tilde{T}$ is an isomorphism, it is linear, and so, using 3.11 of Axler, we have

$$\begin{aligned} 0 &= \tilde{T}(0 + \operatorname{null} T) \\ &= \tilde{T}(b_1(v_1 + \operatorname{null} T) + \cdots + b_n(v_n + \operatorname{null} T)) \\ &= \tilde{T}(b_1(v_1 + \operatorname{null} T)) + \cdots + \tilde{T}(b_n(v_n + \operatorname{null} T)) \\ &= b_1\tilde{T}(v_1 + \operatorname{null} T) + \cdots + b_n\tilde{T}(v_n + \operatorname{null} T) \\ &= b_1Tv_1 + \cdots + b_nTv_n. \end{aligned}$$

Since $Tv_1, \dots, Tv_n$ is a basis of T, it is linearly independent in T. So we have

$$b_1 = 0, \dots, b_n = 0.$$

So $v_1 + \operatorname{null} T, \dots, v_n + \operatorname{null} T$ is linearly independent in $V/(\operatorname{null} T)$. Next, we need to show that $v_1 + \operatorname{null} T, \dots, v_n + \operatorname{null} T$ spans $V/(\operatorname{null} T)$. Since $Tv_1, \dots, Tv_n$ is a basis of range T, it spans range T, meaning that every vector in range T can be written uniquely as

$$Tv = b_1Tv_1 + \cdots + b_nTv_n$$

for some $b_1, \dots, b_n \in \mathbb{F}$. Since $\tilde{T}$ is an isomorphism, it is linear and invertible with inverse $\tilde{T}^{-1}$. For all $v \in V$, we have

$$\begin{aligned} v + \operatorname{null} T &= I_{V/(\operatorname{null} T)}(v + \operatorname{null} T) \\ &= (\tilde{T}^{-1}\tilde{T})(v + \operatorname{null} T) \\ &= \tilde{T}^{-1}(\tilde{T}(v + \operatorname{null} T)) \\ &= \tilde{T}^{-1}(Tv) \\ &= \tilde{T}^{-1}(b_1Tv_1 + \cdots + b_nTv_n) \\ &= b_1\tilde{T}^{-1}(Tv_1) + \cdots + b_n\tilde{T}^{-1}(Tv_n) \\ &= b_1\tilde{T}^{-1}(\tilde{T}(v_1 + \operatorname{null} T)) + \cdots + b_n\tilde{T}^{-1}(\tilde{T}(v_n + \operatorname{null} T)) \\ &= b_1(\tilde{T}^{-1}\tilde{T})(v_1 + \operatorname{null} T) + \cdots + b_n(\tilde{T}^{-1}\tilde{T})(v_n + \operatorname{null} T) \\ &= b_1I_{V/(\operatorname{null} T)}(v_1 + \operatorname{null} T) + \cdots + b_nI_{V/(\operatorname{null} T)}(v + \operatorname{null} T) \\ &= b_1(v_1 + \operatorname{null} T) + \cdots + b_n(v_n + \operatorname{null} T). \end{aligned}$$

So every $v + \operatorname{null} T \in V/(\operatorname{null} T)$ is a linear combination of $v_1 + \operatorname{null} T, \dots, v_n + \operatorname{null} T$. Therefore, we conclude that $v_1 + \operatorname{null} T, \dots, v_n + \operatorname{null} T$ is a basis of $V/(\operatorname{null} T)$.

At this point, we proved that $v_1 + \operatorname{null} T, \dots, v_n + \operatorname{null} T$ is a basis of $V/(\operatorname{null} T)$ if and only if $Tv_1, \dots, Tv_n$ is a basis of range T. Since both of the bases have length n, it follows that we have $\dim(V/(\operatorname{null} T)) = n$ and $\dim \operatorname{range} T = n$. So we conclude

$$\dim(V/(\operatorname{null} T)) = \dim \operatorname{range} T,$$

as desired. □

(5pts) b. Use part (a) of this question and Exercise 3.E.13 of Axler to show that range T is finite-dimensional and that we have

$$\dim V = \dim \operatorname{null} T + \dim \operatorname{range} T.$$

Remark: This is precisely the assertion of the Fundamental Theorem of Linear Maps (3.22 of Axler).

Proof. Because V is finite-dimensional and 3.14 and 3.19 of Axler state that null T and range T are subspaces of V, it follows by 2.26 of Axler that null T and range T are both finite-dimensional. Furthermore, in part (a) we established

$$\dim(V/(\operatorname{null} T)) = \dim(\operatorname{range} T),$$

which implies that $V/(\operatorname{null} T)$ is finite-dimensional. By 2.39 of Axler, there exist a basis $u_1, \ldots, u_m$ of null T and a basis $v_1 + \operatorname{null} T, \ldots, v_n + \operatorname{null} T$ of $V/(\operatorname{null} T)$, which means we have $\dim(\operatorname{null} T) = m$ and $\dim(V/(\operatorname{null} T)) = n$. Consequently, Exercise 3.E.13 of Axler asserts that $u_1, \ldots, u_m, v_1, \ldots, v_n$ is a basis of V, which means we have $\dim V = m + n$. Therefore, we conclude

$$\begin{aligned}\dim V &= m + n \\ &= \dim \operatorname{null} T + \dim(V/(\operatorname{null} T)) \\ &= \dim \operatorname{null} T + \dim \operatorname{range} T\end{aligned}$$

as desired. □