MATH 131: Linear Algebra I
University of California, Riverside
Homework 1 Solutions
July 1, 2019

Solutions to assigned homework problems from *Linear Algebra Done Right* (third edition) by Sheldon Axler

1.A: 4, 5, 6, 9, 10, 11, 12, 13, 15, 16
1.B: 1, 2, 3, 4
1.C: 1, 3, 4, 6, 10, 11, 12, 19, 24

1.A.4. Show that $\alpha + \beta = \beta + \alpha$ for all $\alpha, \beta \in \mathbb{C}$.

Proof. Let $\alpha, \beta \in \mathbb{C}$ be arbitrary. Then there exist $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$ that satisfies $\alpha = \alpha_1 + \alpha_2 i$ and $\beta = \beta_1 + \beta_2 i$. Using the definitions of addition and multiplication in $\mathbb{C}$, we have

$$\begin{aligned}\alpha + \beta &= (\alpha_1 + \alpha_2 i) + (\beta_1 + \beta_2 i)\\ &= (\alpha_1 + \beta_1) + (\alpha_2 + \beta_2)i\end{aligned}$$

and

$$\begin{aligned}\beta + \alpha &= (\beta_1 + \beta_2 i) + (\alpha_1 + \alpha_2 i)\\ &= (\beta + \alpha_1) + (\beta_2 + \alpha_2)i.\end{aligned}$$

Since commutativity holds in $\mathbb{R}$, we have $\alpha_1 + \beta_1 = \beta_1 + \alpha_1$ and $\alpha_2 + \beta_2 = \beta_2 + \alpha_2$. Therefore, we conclude $\alpha + \beta = \beta + \alpha$ for all $\alpha, \beta \in \mathbb{C}$. □

1.A.5. Show that $(\alpha + \beta) + \lambda = \alpha + (\beta + \lambda)$ for all $\alpha, \beta, \lambda \in \mathbb{C}$.

Proof. Let $\alpha, \beta, \lambda \in \mathbb{C}$ be arbitrary. Then there exist $\alpha_1, \alpha_2, \beta_1, \beta_2, \lambda_1, \lambda_2 \in \mathbb{R}$ that satisfies $\alpha = \alpha_1 + \alpha_2 i$, $\beta = \beta_1 + \beta_2 i$, and $\lambda = \lambda_1 + \lambda_2 i$. Using the definitions of addition and multiplication in $\mathbb{C}$, we have

$$\begin{aligned}(\alpha + \beta) + \lambda &= ((\alpha_1 + \alpha_2 i) + (\beta_1 + \beta_2 i)) + (\lambda_1 + \lambda_2 i)\\ &= ((\alpha_1 + \beta_1) + (\alpha_2 + \beta_2)i) + (\lambda_1 + \lambda_2 i)\\ &= ((\alpha_1 + \beta_1) + \lambda_1) + ((\alpha_2 + \beta_2) + \lambda_2)i\end{aligned}$$

and

$$\begin{aligned}\alpha + (\beta + \lambda) &= (\alpha_1 + \alpha_2 i) + ((\beta_1 + \beta_2 i) + (\lambda_1 + \lambda_2 i))\\ &= (\alpha_1 + \alpha_2 i) + ((\beta_1 + \lambda_1) + (\beta_2 + \lambda_2)i)\\ &= (\alpha_1 + (\beta_1 + \lambda_1)) + (\alpha_2 + (\beta_2 + \lambda_2))i.\end{aligned}$$

Since associativity holds in $\mathbb{R}$, we have $(\alpha_1 + \beta_1) + \lambda_1 = \alpha_1 + (\beta_1 + \lambda_1)$ and $(\alpha_2 + \beta_2) + \lambda_2 = \alpha_2 + (\beta_2 + \lambda_2)$. Therefore, we conclude $(\alpha + \beta) + \lambda = \alpha + (\beta + \lambda)$ for all $\alpha, \beta \in \mathbb{C}$. □

1.A.6. Show that $(\alpha\beta)\lambda = \alpha(\beta\lambda)$ for all $\alpha, \beta, \lambda \in \mathbb{C}$.

Proof. Let $\alpha, \beta, \lambda \in \mathbb{C}$ be arbitrary. Then there exist $\alpha_1, \alpha_2, \beta_1, \beta_2, \lambda_1, \lambda_2 \in \mathbb{R}$ that satisfies $\alpha = \alpha_1 + \alpha_2 i$, $\beta = \beta_1 + \beta_2 i$, and $\lambda = \lambda_1 + \lambda_2 i$. Using the definitions of addition and multiplication in $\mathbb{C}$, we have

$$\begin{aligned}(\alpha\beta)\lambda &= ((\alpha_1 + \alpha_2 i)(\beta_1 + \beta_2 i))(\lambda_1 + \lambda_2 i)\\ &= ((\alpha_1\beta_1 - \alpha_2\beta_2) + (\alpha_1\beta_2 + \alpha_2\beta_1)i)(\lambda_1 + \lambda_2 i)\\ &= ((\alpha_1\beta_1 - \alpha_2\beta_2)\lambda_1 - (\alpha_1\beta_2 + \alpha_2\beta_1)\lambda_2) + ((\alpha_1\beta_1 - \alpha_2\beta_2)\lambda_2 + (\alpha_1\beta_2 + \alpha_2\beta_1)\lambda_1)i\end{aligned}$$

and

$$\begin{aligned}\alpha(\beta\lambda) &= (\alpha_1 + \alpha_2 i)((\beta_1 + \beta_2 i)(\lambda_1 + \lambda_2 i))\\ &= (\alpha_1 + \alpha_2 i)((\beta_1\lambda_1 - \beta_2\lambda_2) + (\beta_1\lambda_2 + \beta_2\lambda_1)i)\\ &= (\alpha_1(\beta_1\lambda_1 - \beta_2\lambda_2) - \alpha_2(\beta_1\lambda_2 + \beta_2\lambda_1)) + (\alpha_1(\beta_1\lambda_2 + \beta_2\lambda_1) + \alpha_2(\beta_1\lambda_1 - \beta_2\lambda_2))i.\end{aligned}$$

Since associativity, commutativity, and distributive properties hold in $\mathbb{R}$, we have

$$\begin{aligned}(\alpha_1\beta_1 - \alpha_2\beta_2)\lambda_1 - (\alpha_1\beta_2 + \alpha_2\beta_1)\lambda_2 &= \alpha_1\beta_1\lambda_1 - \alpha_2\beta_2\lambda_1 - \alpha_1\beta_2\lambda_2 - \alpha_2\beta_1\lambda_2\\ &= \alpha_1\beta_1\lambda_1 - \alpha_1\beta_2\lambda_2 - \alpha_2\beta_1\lambda_2 - \alpha_2\beta_2\lambda_1\\ &= \alpha_1(\beta_1\lambda_1 - \beta_2\lambda_2) - \alpha_2(\beta_1\lambda_2 + \beta_2\lambda_1)\end{aligned}$$

and

$$
\begin{aligned}
(\alpha_1\beta_1 - \alpha_2\beta_2)\lambda_2 + (\alpha_1\beta_2 + \alpha_2\beta_1)\lambda_1 &= \alpha_1\beta_1\lambda_2 - \alpha_2\beta_2\lambda_2 + \alpha_1\beta_2\lambda_1 + \alpha_2\beta_1\lambda_1 \\
&= \alpha_1\beta_1\lambda_2 + \alpha_1\beta_2\lambda_1 + \alpha_2\beta_1\lambda_1 - \alpha_2\beta_2\lambda_2 \\
&= \alpha_1(\beta_1\lambda_2 + \beta_2\lambda_1) + \alpha_2(\beta_1\lambda_1 - \beta_2\lambda_2).
\end{aligned}
$$

Therefore, we conclude $(\alpha + \beta) + \lambda = \alpha + (\beta + \lambda)$ for all $\alpha, \beta \in \mathbb{C}$. □

1.A.9. Show that $\lambda(\alpha + \beta) = \lambda\alpha + \lambda\beta$ for all $\lambda, \alpha, \beta \in \mathbb{C}$.

Proof. Let $\alpha, \beta, \lambda \in \mathbb{C}$ be arbitrary. Then there exist $\alpha_1, \alpha_2, \beta_1, \beta_2, \lambda_1, \lambda_2 \in \mathbb{R}$ that satisfies $\alpha = \alpha_1 + \alpha_2 i$, $\beta = \beta_1 + \beta_2 i$, and $\lambda = \lambda_1 + \lambda_2 i$. Using the definitions of addition and multiplication in $\mathbb{C}$, we have

$$
\begin{aligned}
\lambda(\alpha + \beta) &= (\lambda_1 + \lambda_2 i)((\alpha_1 + \alpha_2 i) + (\beta_1 + \beta_2 i)) \\
&= (\lambda_1 + \lambda_2 i)((\alpha_1 + \beta_1) + (\alpha_2 + \beta_2)i) \\
&= (\lambda_1(\alpha_1 + \beta_1) - \lambda_2(\alpha_2 + \beta_2)) + (\lambda_1(\alpha_2 + \beta_2) + \lambda_2(\alpha_1 + \beta_1))i
\end{aligned}
$$

and

$$
\begin{aligned}
\lambda\alpha + \lambda\beta &= (\lambda_1 + \lambda_2 i)(\alpha_1 + \alpha_2 i) + (\lambda_1 + \lambda_2 i)(\beta_1 + \beta_2 i) \\
&= ((\lambda_1\alpha_1 - \lambda_2\alpha_2) + (\lambda_1\alpha_2 + \lambda_2\alpha_1)i) + ((\lambda_1\beta_1 - \lambda_2\beta_2) + (\lambda_1\beta_2 + \lambda_2\beta_1)i) \\
&= ((\lambda_1\alpha_1 - \lambda_2\alpha_2) + (\lambda_1\beta_1 - \lambda_2\beta_2)) + ((\lambda_1\alpha_2 + \lambda_2\alpha_1) + (\lambda_1\beta_2 + \lambda_2\beta_1))i.
\end{aligned}
$$

Since associativity, commutativity, and distributive properties hold in $\mathbb{R}$, we have

$$
\begin{aligned}
\lambda_1(\alpha_1 + \beta_1) - \lambda_2(\alpha_2 + \beta_2) &= \lambda_1\alpha_1 + \lambda_1\beta_1 - \lambda_2\alpha_2 - \lambda_2\beta_2 \\
&= (\lambda_1\alpha_1 - \lambda_2\alpha_2) + (\lambda_1\beta_1 + \lambda_2\beta_2)
\end{aligned}
$$

and

$$
\begin{aligned}
\lambda_1(\alpha_2 + \beta_2) + \lambda_2(\alpha_1 + \beta_1) &= \lambda_1\alpha_2 + \lambda_1\beta_2 + \lambda_2\alpha_1 + \lambda_2\beta_1 \\
&= (\lambda_1\alpha_2 + \lambda_2\alpha_1) + (\lambda_1\beta_2 + \lambda_2\beta_1).
\end{aligned}
$$

Therefore, we conclude $\lambda(\alpha + \beta) = \lambda\alpha + \lambda\beta$ for all $\lambda, \alpha, \beta \in \mathbb{C}$. □

1.A.10. Find $x \in \mathbb{R}^4$ such that

$$(4, -3, 1, 7) + 2x = (5, 9, -6, 8).$$

Proof. Since we have $x \in \mathbb{R}^4$, it is a list of length 4, and so we can write $x = (x_1, x_2, x_3, x_4)$. So we can rewrite the desired equation as

$$(4, -3, 1, 7) + 2(x_1, x_2, x_3, x_4) = (5, 9, -6, 8).$$

Applying addition and scalar multiplication in $\mathbb{F}^n$ to the left-hand side of the above equation, we get

$$(4 + 2x_1, -3 + 2x_2, 1 + 2x_3, 7 + 2x_4) = (5, 9, -6, 8).$$

Equating the coordinates of the vectors on both sides of the above equation, we obtain four separate equations

$$4 + 2x_1 = 5, -3 + 2x_2 = 9, 1 + 2x_3 = -6, 7 + 2x_4 = 8,$$

from which we can solve individually to obtain

$$x_1 = \frac{1}{2}, x_2 = 6, x_3 = -\frac{7}{2}, x_4 = \frac{1}{2}.$$

Therefore, we have $x = (x_1, x_2, x_3, x_4) = (\frac{1}{2}, 6, -\frac{7}{2}, \frac{1}{2})$. □

1.A.11. Explain why there does not exist $\lambda \in \mathbb{C}$ such that

$$\lambda(2 - 3i, 5 + 4i, -6 + 7i) = (12 - 5i, 7 + 22i, -32 - 9i).$$

Proof. Suppose by contradiction that there exists $\lambda \in \mathbb{C}$ that satisfies

$$\lambda(2 - 3i, 5 + 4i, -6 + 7i) = (12 - 5i, 7 + 22i, -32 - 9i).$$

Applying scalar multiplication in $\mathbb{F}^n$ to the left-hand side of the above equation, we get

$$(\lambda(2 - 3i), \lambda(5 + 4i), \lambda(-6 + 7i)) = (12 - 5i, 7 + 22i, -32 - 9i).$$

We can choose any two of the three coordinates to find our desired contradiction. Let us, for instance, take the first and second coordinates:

$$\lambda(2-3i) = 12-5i$$

and

$$\lambda(5+4i) = 7+22i.$$

So, from the first coordinates, we get

$$\begin{aligned} 13\lambda &= ((2-3i)(2+3i))\lambda \\ &= (\lambda(2-3i))(2+3i) \\ &= (12-5i)(2+3i) \\ &= 39+26i \end{aligned}$$

and, from the second coordinates, we get

$$\begin{aligned} 41\lambda &= ((5+4i)(5-4i))\lambda \\ &= (\lambda(5+4i))(5-4i) \\ &= (7+22i)(5+4i) \\ &= -53+138i. \end{aligned}$$

So we get $\lambda = 3+2i$ and $\lambda = -\frac{53}{41} + \frac{138}{41}i$ simultaneously, which is a contradiction. □

1.A.12. Show that $(x+y)+z = x+(y+z)$ for all $x, y, z \in \mathbb{F}^n$.

Proof. Since we have $x, y, z \in \mathbb{F}^n$, they are all lists of length n, which means we can write

$$x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n), z = (z_1, \ldots, z_n).$$

So, for all $x, y, z \in \mathbb{F}^n$, we have

$$\begin{aligned} (x+y)+z &= ((x_1, \ldots, x_n) + (y_1, \ldots, y_n)) + (z_1, \ldots, z_n) \\ &= (x_1+y_1, \ldots, x_n+y_n) + (z_1, \ldots, +z_n) \\ &= ((x_1+y_1)+z_1, \ldots, (x_n+y_n)+z_n) \end{aligned}$$

and

$$\begin{aligned} x+(y+z) &= (x_1, \ldots, x_n) + ((y_1, \ldots, y_n) + (z_1, \ldots, z_n)) \\ &= (x_1, \ldots, x_n) + (y_1+z_1, \ldots, y_n+z_n) \\ &= (x_1+(y_1+z_1), \ldots, x_n+(y_n+z_n)). \end{aligned}$$

Since associativity holds in $\mathbb{F}$, we have $(x_i+y_i)+z_i = x_i+(y_i+z_i)$ for all $i = 1, \ldots, n$. So we conclude $(x+y)+z = x+(y+z)$ for all $x, y, z \in \mathbb{F}^n$. □

1.A.13. Show that $(ab)x = a(bx)$ for all $x \in \mathbb{F}^n$ and for all $a, b \in \mathbb{F}$.

Proof. Since we have $x \in \mathbb{F}^n$, it is a list of length n, which means we can write $x = (x_1, \ldots, x_n)$. So, for all $a, b \in \mathbb{F}$ and for all $x \in \mathbb{F}^n$, we have

$$\begin{aligned} (ab)x &= (ab)(x_1, \ldots, x_n) \\ &= ((ab)x_1, \ldots, (ab)x_n) \end{aligned}$$

and

$$\begin{aligned} a(bx) &= a(b(x_1, \ldots, x_n)) \\ &= a(bx_1, \ldots, bx_n) \\ &= (a(bx_1), \ldots, a(bx_n)). \end{aligned}$$

Since associativity holds in $\mathbb{F}$, we have $(ab)x_i = a(bx_i)$ for all $i = 1, \ldots, n$. So we conclude $(ab)x = a(bx)$ for all $a, b \in \mathbb{F}$ and for all $x \in \mathbb{F}^n$. □

1.A.15. Show that $\lambda(x+y) = \lambda x + \lambda y$ for all $\lambda \in \mathbb{F}$ and for all $x, y \in \mathbb{F}^n$.

Proof. Since we have $x, y \in \mathbb{F}^n$, they are lists of length n, which means we can write $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n)$. So, for all $\lambda \in \mathbb{F}$ and for all $x, y \in \mathbb{F}^n$, we have

$$\begin{aligned}\lambda(x+y) &= \lambda((x_1,\ldots,x_n)+(y_1,\ldots,y_n))\\ &= \lambda(x_1+y_1,\ldots,x_n+y_n)\\ &= (\lambda(x_1+y_1),\ldots,\lambda(x_n+y_n))\end{aligned}$$

and

$$\begin{aligned}\lambda x+\lambda y &= \lambda(x_1,\ldots,x_n)+\lambda(y_1,\ldots,y_n)\\ &= (\lambda x_1,\ldots,\lambda x_n)+(\lambda y_1,\ldots,\lambda y_n)\\ &= (\lambda x_1+\lambda y_1,\ldots,\lambda x_n+\lambda y_n).\end{aligned}$$

Since distributive properties hold in $\mathbb{F}$, we have $\lambda(x_i + y_i) = \lambda x_i + \lambda y_i$ for all $i = 1, \ldots, n$. So we conclude $\lambda(x + y) = \lambda x + \lambda y$ for all $\lambda \in \mathbb{F}$ and for all $x, y \in \mathbb{F}^n$. □

1.A.16. Show that $(a + b)x = ax + bx$ for all $a, b \in \mathbb{F}$ and for all $x \in \mathbb{F}^n$.

Proof. Since we have $x \in \mathbb{F}^n$, it is a list of length n, which means we can write $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n)$. So, for all $a, b \in \mathbb{F}$ and for all $x \in \mathbb{F}^n$, we have

$$\begin{aligned}(a+b)x &= (a+b)(x_1,\ldots,x_n)\\ &= ((a+b)x_1,\ldots,(a+b)x_n)\end{aligned}$$

and

$$\begin{aligned}ax+bx &= a(x_1,\ldots,x_n)+b(x_1,\ldots,x_n)\\ &= (ax_1,\ldots,ax_n)+(bx_1,\ldots,bx_n)\\ &= (ax_1+bx_1,\ldots,ax_n+bx_n).\end{aligned}$$

Since distributive properties hold in $\mathbb{F}$, we have $(a + b)x_i = ax_i + bx_i$ for all $i = 1, \ldots, n$. So we conclude $(a + b)x = ax + bx$ for all $a, b \in \mathbb{F}$ and for all $x \in \mathbb{F}^n$. □

1.B.1. Prove that $-(-v) = v$ for all $v \in V$.

Proof. Since V is a vector space, we can use scalar multiplication on V to get $-v \in V$ for all $v \in V$. We also have that elements in V are commutative: for all $v, w \in V$ we have $v + w = w + v$. So, for all $v \in V$, we have

$$\begin{aligned}(-v)+v &= v+(-v)\\ &= v-v\\ &= 0,\end{aligned}$$

which means that v is an additive inverse of $-v$. Also, for all $v \in V$, we have

$$\begin{aligned}(-v)+(-(-v)) &= (-v)-(-v)\\ &= 0,\end{aligned}$$

which means that $-(-v)$ is an additive inverse of $-v$. However, 1.26 of Axler asserts that the addivitve inverse is unique. Since we established already that v and $-(-v)$ are both additive inverses of V, we conclude from 12.6 of Axler that we must have $-(-v) = v$ for all $v \in V$. □

1.B.2. Suppose $a \in \mathbb{F}$ and $v \in V$ satisfy $av = 0$. Prove that we have $a = 0$ or $v = 0$.

Proof. We will argue this in two separate cases: $a = 0$ and $a \neq 0$.

- If $a = 0$, then certainly we have $a \in F$. also, according to 1.29 of Axler, for all $v \in V$, we have

$$\begin{aligned}av &= 0v\\ &= 0.\end{aligned}$$

  So $a = 0$ satisfies the hypotheses. At the same time $a = 0$ is already one of the statements of the conclusion: $a = 0$ or $v = 0$. So, for this case, we are done; there is nothing to prove.

- If $a \neq 0$, then we will prove $v = 0$. Since we assume $a \neq 0$, there exists the multiplicative inverse $\frac{1}{a}$ which satisfies $a(\frac{1}{a}) = 1$. Since $av = 0$, we have

$$\begin{aligned} v &= 1v \\ &= \left(a\left(\frac{1}{a}\right)\right)v \\ &= \left(\left(\frac{1}{a}\right)a\right)v \\ &= \left(\frac{1}{a}\right)(av) \\ &= \left(\frac{1}{a}\right)(0) \\ &= 0, \end{aligned}$$

  as we claimed.

Therefore, in either case, we have either $a = 0$ or $v = 0$. □

1.B.3. Suppose we have $v, w \in V$. Explain why there exists a unique $x \in V$ such that $v + 3x = w$.

Proof. Let $x = \frac{1}{3}w - \frac{1}{3}v$. Since V is a vector space and $\frac{1}{3} \in \mathbb{F}$, for all $v, w \in V$, we can apply scalar multiplication on V to conclude $x \in V$. Moreover, we have

$$\begin{aligned} v + 3x &= v + 3\left(\frac{1}{3}w - \frac{1}{3}v\right) \\ &= v + 3\left(\frac{1}{3}\right)w - 3\left(\frac{1}{3}\right)v \\ &= v + w - v \\ &= w. \end{aligned}$$

This completes our proof that $x \in V$ exists and satisfies $v + 3x = w$. At this point, we are left to prove that $x \in V$ is unique. Suppose $y \in V$ satisfies $v + 3y = w$. Then we have

$$\begin{aligned} 0 &= w - w \\ &= (v + 3x) - (v + 3y) \\ &= v + 3x - v - 3y \\ &= v - v + 3x - 3y \\ &= 3x - 3y \\ &= 3(x - y). \end{aligned}$$

By Exercise 1.B.2 of Axler, either $3 = 0$ or $x - y = 0$. Since $3 = 0$ is a false statement, we must have $x - y = 0$, or equivalently, $x = y$, which means $x \in V$ satisfying $v + 3x = w$ is unique. □

1.B.4. The empty set is not a vector space. The empty set fails to satisfy only one of the requirements listed in 1.19 of Axler. Which one?

Proof. The additive identity is the only property of a vector space that asserts an existence of an element $0 \in V$ without any conditions. Since the empty set does not contain any elements, it cannot contain for example the zero vector. So the empty set fails to satisfy the property of additive identity in a vector space. However, all the other properties of a vector space hold vacuously for an empty set. By "vacuous", we mean that there are no vectors to work with, so we cannot possibly disprove the other properties and render the conclusion false. We must therefore assume that the conclusion is true. □

1.C.1. For each of the following subsets of $\mathbb{F}^3$, determine whether it is a subspace of $\mathbb{F}^3$. If so, prove it. If not, give a counterexample to show some property of a subspace that is not satisfied.

(a) $\{(x_1, x_2, x_3) \in \mathbb{F}^3 : x_1 + 2x_2 + 3x_3 = 0\}$;

Proof. We will prove that $U_1 = \{(x_1, x_2, x_3) \in \mathbb{F}^3 : x_1 + 2x_2 + 3x_3 = 0\}$ is a subspace of $\mathbb{F}^3$.

- Additive identity: Since $(0) + 2(0) + 3(0) = 0$, we have $(0, 0, 0) \in U_1$.

- Closed under addition: Suppose we have $(x_1, x_2, x_3), (y_1, y_2, y_3) \in U_1$. Then we have $x_1 + 2x_2 + 3x_3 = 0$ and $y_1 + 2y_2 + 3y_3 = 0$. These imply

$$\begin{aligned}(x_1 + y_1) + 2(x_2 + y_2) + 3(x_3 + y_3) &= (x_1 + 2x_2 + 3x_3) + (y_1 + 2y_2 + 3y_3)\\ &= 0 + 0\\ &= 0.\end{aligned}$$

  So we conclude $(x_1, x_2, x_3) + (y_1, y_2, y_3) = (x_1 + y_1, x_2 + y_2, x_3 + y_3) \in U_1$.

- Closed under scalar multiplication: Suppose we have $\lambda \in \mathbb{F}$ and $(x_1, x_2, x_3) \in U_1$. Then we have $x_1 + 2x_2 + 3x_3 = 0$. This implies

$$\begin{aligned}(\lambda x_1) + 2(\lambda x_2) + 3(\lambda x_3) &= \lambda(x_1 + 2x_2 + 3x_3)\\ &= \lambda \cdot 0\\ &= 0.\end{aligned}$$

  So we conclude $\lambda(x_1, x_2, x_3) = (\lambda x_1, \lambda x_2, \lambda x_3) \in U_1$.

Since we satisfied all the properties of a subspace, we conclude that U_1 is a subspace of $\mathbb{F}^3$. □

(b) $\{(x_1, x_2, x_3) \in \mathbb{F}^3 : x_1 + 2x_2 + 3x_3 = 4\}$;

Proof. We will prove that $U_2 = \{(x_1, x_2, x_3) \in \mathbb{F}^3 : x_1 + 2x_2 + 3x_3 = 4\}$ is not a subspace of $\mathbb{F}^3$.

- Additive identity is not satisfied. Since we have $0 + 2(0) + 3(0) = 0 \neq 4$, we conclude $(0, 0, 0) \notin U_2$.

Since we showed that one of the properties of a subspace is not satisfied, we conclude that U_2 is not a subspace of $\mathbb{F}^3$. □

(c) $\{(x_1, x_2, x_3) \in \mathbb{F}^3 : x_1x_2x_3 = 0\}$;

Proof. We will prove that $U_3 = \{(x_1, x_2, x_3) \in \mathbb{F}^3 : x_1x_2x_3 = 0\}$ is not a subspace of $\mathbb{F}^3$.

- Closed under addition is not satisfied. Let $(x_1, x_2, x_3) = (0, 1, 0), (y_1, y_2, y_3) = (1, 0, 1) \in \mathbb{F}^3$. Then we have $x_1x_2x_3 = (0)(1)(0)$ and $y_1y_2y_3 = (1)(0)(1) = 0$, which means we have $(x_1, x_2, x_3), (y_1, y_2, y_3) \in U_3$. But these imply

$$\begin{aligned}(x_1 + y_1)(x_2 + y_2)(x_3 + y_3) &= (0 + 1)(1 + 0)(0 + 1)\\ &= (1)(1)(1)\\ &= 1\\ &\neq 0,\end{aligned}$$

  which means we have $(x_1, x_2, x_3) + (y_1, y_2, y_3) = (x_1 + y_1, x_2 + y_2, x_3 + y_3) \notin U_3$.

Since we showed that one of the properties of a subspace is not satisfied, we conclude that U_3 is not a subspace of $\mathbb{F}^3$. □

(d) $\{(x_1, x_2, x_3) \in \mathbb{F}^3 : x_1 = 5x_3\}$.

Proof. We will prove that $U_4 = \{(x_1, x_2, x_3) \in \mathbb{F}^3 : x_1 = 5x_3\}$ is a subspace of $\mathbb{F}^3$.

- Additive identity: Since $(0) = 5(0)$, we have $(0, 0, 0) \in U_4$.
- Closed under addition: Suppose we have $(x_1, x_2, x_3), (y_1, y_2, y_3) \in U_4$. Then we have $x_1 = 5x_3$ and $y_1 = 5y_3$. These imply

$$\begin{aligned}x_1 + y_1 &= 5x_3 + 5y_3\\ &= 5(x_3 + y_3).\end{aligned}$$

  So we conclude $(x_1, x_2, x_3) + (y_1, y_2, y_3) = (x_1 + y_1, x_2 + y_2, x_3 + y_3) \in U_1$.

- Closed under scalar multiplication: Suppose we have $\lambda \in \mathbb{F}$ and $(x_1, x_2, x_3) \in U_1$. Then we have $x_1 = 5x_3$. This implies

$$\begin{aligned}\lambda x_1 &= \lambda(5x_3)\\ &= 5(\lambda x_3).\end{aligned}$$

  So we conclude $\lambda(x_1, x_2, x_3) = (\lambda x_1, \lambda x_2, \lambda x_3) \in U_1$.

Since we satisfied all the properties of a subspace, we conclude that U_4 is a subspace of $\mathbb{F}^3$. □

1.C.3. Show that the set of differentiable real-valued functions f on the interval $(-4, 4)$ such that $f'(-1) = 3f(2)$ is a subspace of $\mathbb{R}^{(-4,4)}$.

Proof. Let U be the set of differentiable real-valued functions f on the interval $(-4, 4)$ such that $f'(-1) = 3f(2)$ is a subspace of $\mathbb{R}^{(-4,4)}$. We will prove that U is a subspace of $\mathbb{F}^3$.

- Additive identity: Note that, for all $x \in (-4, 4)$ the derivative of the zero function $0(x) = 0$ is still the zero function. In other words, we have $0'(x) = 0(x) = 0$ for all $x \in (-4, 4)$. In particular, we have $0'(-1) = 0 = 3 \cdot 0 = 30(2)$, and so we conclude $0 \in \mathbb{R}^{(-4,4)}$
- Closed under addition: Let $f, g \in U$ be arbitrary. Then we have $f'(-1) = 3f(2)$ and $g'(-1) = 3g(2)$, which imply

$$\begin{aligned}(f+g)'(-1) &= (f'+g')(-1)\\ &= f'(-1) + g'(-1)\\ &= 3f(2) + 3g(2)\\ &= 3(f(2) + g(2))\\ &= 3(f+g)(2).\end{aligned}$$

  So we conclude $f + g \in U$.
- Closed under scalar multiplication: Let $\lambda \in \mathbb{F}$ and $f \in U$ be arbitrary. Then we have $f'(-1) = 3f(2)$, which imply

$$\begin{aligned}(\lambda f)'(-1) &= (\lambda f')(-1)\\ &= \lambda f'(-1)\\ &= \lambda(3f(2))\\ &= 3\lambda f(2)\\ &= 3(\lambda f)(2).\end{aligned}$$

  So we conclude $\lambda f \in U$.

Since we satisfied all the properties of a subspace, we conclude that U is a subspace of $\mathbb{R}^{(-4,4)}$. □

1.C.4. Suppose $b \in \mathbb{R}$. Show that the set of continuous real-valued functions f on the interval $[0, 1]$ such that $\int_0^1 f(x)\,dx = b$ is a subspace of $\mathbb{R}^{[0,1]}$ if and only if $b = 0$.

Proof. For brevity, let U be the set of continuous real-valued functions f on the interval $[0, 1]$ such that $\int_0^1 f(x)\,dx = b$ is a subspace of $\mathbb{R}^{[0,1]}$.

Forward direction: If U is a subspace of $\mathbb{R}^{[0,1]}$, then $b = 0$. Suppose U is a subspace of $\mathbb{R}^{[0,1]}$ and $f \in U$. Then it contains the additive identity; namely, it contains the zero function 0. So, for all $x \in [0, 1]$, we have

$$\begin{aligned}b &= \int_0^1 f(x)\,dx\\ &= \int_0^1 0(x)\,dx\\ &= \int_0^1 0\,dx\\ &= 0,\end{aligned}$$

as desired.

Backward direction: If $b = 0$, then U is a subspace of $\mathbb{R}^{[0,1]}$. We will prove that U is a subspace of $\mathbb{R}^{[0,1]}$.

- Additive identity: Note that we have

$$\begin{aligned}\int_0^1 0(x)\,dx &= \int_0^1 0\,dx\\ &= 0\\ &= b,\end{aligned}$$

  and so we have $0 \in U$.
- Closed under addition: Let $f, g \in U$ be arbitrary. Then they satisfy $\int_0^1 f(x)\,dx = b = 0$ and $\int_0^1 g(x)\,dx = b = 0$. So we have

$$\begin{aligned}\int_0^1 (f+g)(x)\,dx &= \int_0^1 f(x) + g(x)\,dx\\ &= \int_0^1 f(x)\,dx + \int_0^1 g(x)\,dx\\ &= b + b\\ &= 0 + 0\\ &= 0,\end{aligned}$$

  and so we have $f + g \in U$.

- Closed under scalar multiplication: Let $\lambda \in \mathbb{F}$ and $f \in U$ be arbitrary. Then it satisfies $\int_0^1 f(x)\,dx = b = 0$. So we have

$$\begin{aligned}\int_0^1 (\lambda f)(x)\,dx &= \int_0^1 \lambda f(x)\,dx \\ &= \lambda \int_0^1 f(x)\,dx \\ &= \lambda b \\ &= \lambda \cdot 0 \\ &= 0,\end{aligned}$$

  and so we have $\lambda f \in U$.

Since we satisfied all the properties of a subspace, we conclude that U is a subspace of $\mathbb{R}^{[0,1]}$. □

1.C.6. (a) Is $\{(a,b,c) \in \mathbb{R}^3 : a^3 = b^3\}$ a subspace of $\mathbb{R}^3$?

Proof. Let $U_1 = \{(a,b,c) \in \mathbb{R}^3 : a^3 = b^3\}$. First, we will prove that, for all $a, b \in \mathbb{R}$, we have $a^3 = b^3$ if and only if $a = b$.
Forward direction: If $a^3 = b^3$, then $a = b$. Suppose $a, b \in \mathbb{R}$ satisfies $a^3 = b^3$. If $a \neq 0$ or $b \neq 0$, then

$$\begin{aligned}a^2 + ab + b^2 &= \left(a^2 + ab + \frac{1}{4}b^2\right) - \frac{1}{4}b^2 + b^2 \\ &= \left(a + \frac{1}{2}b\right)^2 + \frac{3}{4}b^2 \\ &> 0.\end{aligned}$$

Therefore, if $a \neq 0$ or $b \neq 0$, then $a^3 = b^3$ implies

$$\begin{aligned}0 &= a^3 - b^3 \\ &= (a-b)(a^2 + ab + b^2),\end{aligned}$$

from which, according to Exercise 1.B.2 of Axler, we get $a - b = 0$, or $a = b$, since we have $a^2 + ab + b^2 > 0$. On the other hand, if $a = 0$ and $b = 0$, then of course we have $a = b$.
Backward direction: If $a = b$, then $a^3 = b^3$. Suppose $a = b$. Then $a - b = 0$, and so we have

$$\begin{aligned}0 &= 0^3 \\ &= (a-b)^3 \\ &= a^3 - 3a^2b + 3ab^2 - b^3 \\ &= a^3 - 3a^2a + 3bb^2 - b^3 \\ &= a^3 - 3a^3 + 3b^3 - b^3 \\ &= -2a^3 + 2b^3 \\ &= 2(-a^3) + 2b^3 \\ &= 2(-a^3 + b^3),\end{aligned}$$

and so by Exercise 1.B.2 of Axler, we get $-a^3 + b^3 = 0$, or $a^3 = b^3$.
So we proved that, for all $a, b \in \mathbb{R}$, we have $a^3 = b^3$ if and only if $a = b$. This means we can rewrite our subset of $\mathbb{R}^3$ as

$$\begin{aligned}U_1 &= \{(a,b,c) \in \mathbb{R}^3 : a^3 = b^3\} \\ &= \{(a,b,c) \in \mathbb{R}^3 : a = b\}.\end{aligned}$$

Next, we will prove that $U_1 = \{(a,b,c) \in \mathbb{R}^3 : a = b\}$ is a subspace of $\mathbb{R}^3$.

- Additive identity: Note that we have $0 = 0$, and so we have $(0,0,0) \in U_1$.
- Closed under addition: Let $(a,b,c),(d,e,f) \in U_1$ be arbitrary. Then $a,b,c,d,e,f \in \mathbb{R}$ satisfy $a = b$ and $d = e$. So we have $a + d = b + e$, and so we have $(a,b,c) + (d,e,f) = (a+d, b+e, c+f) \in U$.
- Closed under scalar multiplication: Let $\lambda \in \mathbb{F}$ and $(a,b,c) \in U_1$ be arbitrary. Then $a,b,c \in \mathbb{R}$ satisfies $a = b$. So we have $\lambda a = \lambda b$, and so we have $\lambda(a,b,c) = (\lambda a, \lambda b, \lambda c) \in U$.

Therefore, U_1 is a subspace of $\mathbb{R}^3$. □

(b) Is $\{(a,b,c) \in \mathbb{C}^3 : a^3 = b^3\}$ a subspace of $\mathbb{C}^3$?

Proof. Let $U_2 = \{(a, b, c) \in \mathbb{C}^3 : a^3 = b^3\}$. We will give a counterexample to show that U_2 is not a subspace of $\mathbb{C}^3$. Let $(a, b, c) = (\frac{1}{2} + \frac{\sqrt{3}}{2}i, 1, 0), (d, e, f) = (\frac{1}{2} - \frac{\sqrt{3}}{2}i, 1, 0) \in U_2$. Then they satisfy

$$\begin{aligned} a^3 &= \left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)^3 \\ &= 1 \\ &= b^3 \end{aligned}$$

and

$$\begin{aligned} d^3 &= \left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)^3 \\ &= 1 \\ &= e^3. \end{aligned}$$

However, we have

$$\begin{aligned} (a + d)^3 &= \left(\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) + \left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)\right)^3 \\ &= 1^3 \\ &= 1 \end{aligned}$$

and

$$\begin{aligned} (b + e)^3 &= (1 + 1)^3 \\ &= 2^3 \\ &= 8. \end{aligned}$$

Since we have $1 \neq 8$, we conclude $(a + d)^3 \neq (b + e)^3$. So we get $(a, b, c) + (d, e, f) = (a + d, b + e, c + f) \notin U_2$, meaning that U_2 is not closed under addition. Therefore, U_2 is not a subspace of $\mathbb{C}^3$. □

1.C.10. Suppose U_1 and U_2 are subspaces of V. Prove that the intersection $U_1 \cap U_2$ is a subspace of V.

Proof. Let $\lambda \in \mathbb{F}$ and $u, w \in U_1 \cap U_2$ be arbitrary; this means we will argue for all $\lambda \in \mathbb{F}$ and for all $u, w \in U_1 \cap U_2$.

- Additive identity: Since U_1 and U_2 are subspaces of V, we have $0 \in U_1$ and $0 \in U_2$. Therefore, we have $0 \in U_1 \cap U_2$.
- Closed under addition: We have $u, w \in U_1 \cap U_2$, which means we have $u, w \in U_1$ and $u, w \in U_2$. Since U_1 and U_2 are subspaces of V, they are closed under addition, and so we have $u + w \in U_1$ and $u + w \in U_2$. Therefore, we have $u + w \in U_1 \cap U_2$.
- Closed under scalar multiplication: We have $u \in U_1 \cap U_2$, which means we have $u \in U_1$ and $u \in U_2$. Since U_1 and U_2 are subspaces of V, they are closed under scalar multiplication, and so we have $\lambda u \in U_1$ and $\lambda u \in U_2$. Therefore, we have $\lambda u \in U_1 \cap U_2$.

Since we satisfied all the properties of a subspace, we conclude that U is a subspace of $\mathbb{C}^{\mathbb{R}}$. □

1.C.11. Prove that the intersection of every collection of subspaces of V is a subspace of V.

Proof. Let A be a collection of indices and let $a \in A$ be an arbitrary index. Let $\{U_a\}_a$ be an arbitrary collection of subspaces U_a in V. Let $\lambda \in \mathbb{F}$ and $u_a, w_a \in \bigcap_{a \in A} U_a$ be arbitrary; this means we will argue for all $\lambda \in \mathbb{F}$ and for all $u_a, w_a \in \bigcap_{a \in A} U_a$.

- Additive identity: Since every U_a is a subspace of V, we have $0 \in U_a$ for all $a \in A$. Therefore, we have $0 \in \bigcap_{a \in A} U_a$.
- Closed under addition: We have $u_a, w_a \in \bigcap_{a \in A} U_a$, which means we have $u_a, w_a \in U_a$ for all $a \in A$. Since every U_a is a subspace of V, each one is closed under addition, and so we have and $u_a + w_a \in U_a$ for all $a \in A$. Therefore, we have $u_a + w_a \in \bigcap_{a \in A} U_a$.
- Closed under scalar multiplication: We have $u_a \in U_a$, which means we have $u_a \in U_a$ for all $a \in A$. Since every U_a is a subspace of V, each one is closed under scalar multiplication, and so we have $\lambda u_a \in U_a$ for all $a \in A$. Therefore, we have $\lambda u_a \in \bigcap_{a \in A} U_a$.

Since we satisfied all the properties of a subspace, we conclude that U is a subspace of $\mathbb{C}^{\mathbb{R}}$. □

1.C.12. Prove that the union of two subspaces of V is a subspace of V if and only if one of the subspaces is contained in the other.

Proof. Let U_1, U_2 be subspaces of V. So we will prove that $U_1 \cup U_2$ is a subspace of V if and only if we have $U_1 \subset U_2$ or $U_2 \subset U_1$.

Forward direction: If $U_1 \cup U_2$ is a subspace of V, then we have $U_1 \subset U_2$ or $U_2 \subset U_1$. Suppose $U_1 \cup U_2$ is a subspace of V. Suppose by contradiction we have instead $U_1 \not\subset U_2$ and $U_2 \not\subset U_1$ (this is the negation of the statement $U_1 \subset U_2$ or $U_2 \subset U_1$). Then $U_1 \setminus U_2$ and $U_2 \setminus U_1$ are nonempty, which means there exist vectors $u_1 \in U_1 \setminus U_2$ and $u_2 \in U_2 \setminus U_1$. We observe the containments $U_1 \setminus U_2, U_2 \setminus U_1 \subset U_1 \cup U_2$, which means we have in fact $u_1, u_2 \in U_1 \cup U_2$. Since $U_1 \cup U_2$ is a subspace of V, it is closed under addition, and so we have $u_1 + u_2 \in U_1 \cup U_2$, which also means we have $u_1 + u_2 \in U_1$ or $u_1 + u_2 \in U_2$. Also, as subspaces, U_1 and U_2 are closed under scalar multiplication, which means we have $-u_1 \in U_1$ and $-u_2 \in U_2$. So we obtain $u_2 = (u_1 + u_2) - u_1 \in U_1$ and $u_1 = (u_1 + u_2) - u_2 \in U_2$. Since we also have from earlier $u_1 \in U_1 \setminus U_2$ and $u_2 \in U_2 \setminus U_1$, we conclude $u_1 \in (U_1 \setminus U_2) \cap U_2$ and $u_2 \in (U_2 \setminus U_1) \cap U_1$. In other words, there exist elements in $(U_1 \setminus U_2) \cap U_2$ and $(U_2 \setminus U_1) \cap U_1$, but this contradicts the fact that these sets are empty.

Backward direction: If we have $U_1 \subset U_2$ or $U_2 \subset U_1$, then $U_1 \cup U_2$ is a subspace of V. Suppose we have $U_1 \subset U_2$ or $U_2 \subset U_1$. If $U_1 \subset U_2$, then we will prove $U_1 \cup U_2 = U_2$. Suppose we have $u \in U_1 \cup U_2$. Then $u \in U_1$ or $u \in U_2$. But $U_1 \subset U_2$ implies $u \in U_2$. So in either case, we have $u \in U_2$. Therefore, we get $U_1 \cup U_2 \subset U_2$. Conversely, suppose we have $u \in U_2$. Then we have $u \in U_1$ or $u \in U_2$. So we get $u \in U_1 \cup U_2$, and so $U_2 \subset U_1 \cup U_2$. Therefore, we conclude the set equality $U_1 \cup U_2 = U_2$. If $U_2 \subset U_1$, then we can simply interchange the roles of U_1 and U_2 in the previous statement "If $U_1 \subset U_2$, then $U_1 \cup U_2 = U_2$" in order to establish the desired statement "If $U_2 \subset U_1$, then $U_1 \cup U_2 = U_1$". So we have established: if $U_1 \subset U_2$ or $U_2 \subset U_1$, then we have $U_1 \cup U_2 = U_2$ or $U_1 \cup U_2 = U_1$, respectively. In either case, as we assumed in the premises that U_1 and U_2 are subspaces of V, we conclude that $U_1 \cup U_2$ is a subspace of V. □

1.C.19. Prove or give a counterexample: if U_1, U_2, W are subspaces of V such that

$$U_1 + W = U_2 + W,$$

then $U_1 = U_2$.

Proof. We will give a counterexample to prove that this statement is false. Let $V = \mathbb{R}^2$ be the vector space, and consider the subsets $U_1 = \{(x_1, 0) : x_1 \in \mathbb{R}\}$, $U_2 = \{(x_1, x_1) : x_1 \in \mathbb{R}\}$, and $W = \{(0, x_2) : x_2 \in \mathbb{R}\}$. We must prove that U_1, U_2, W are subspaces of $\mathbb{R}^2$.

- Additive identity: Note that we have $0 \in \mathbb{R}$, which means $(0,0) \in U_1$, $(0,0) \in U_2$, and $(0,0) \in W$. Therefore, U_1, U_w, W all contain the additive identity.
- Closed under addition: Suppose that we have $(x_1, 0), (y_1, 0) \in U_1$, $(x_1, x_1), (y_1, y_1) \in U_2$, and $(0, x_2), (0, y_2) \in W$ for all $x_1, x_2, y_1, y_2 \in \mathbb{R}$. Then, since we have $x_1 + y_1, x_2 + y_2 \in \mathbb{R}$, it follows that we have

$$(x_1, 0) + (y_1, 0) = (x_1 + y_1, 0) \in U_1,$$
$$(x_1, x_1) + (y_1, y_1) = (x_1 + y_1, x_1 + y_1) \in U_2,$$
$$(0, x_2) + (0, y_2) = (0, x_2 + y_2) \in W.$$

  Therefore, U_1, U_2, W are all closed under addition.
- Closed under scalar multiplication: Let $\lambda \in \mathbb{F}$ be arbitrary. Suppose that we have $(x_1, 0) \in U_1$, $(x_1, x_1) \in U_2$, and $(0, x_2) \in W$. Then, since $\lambda x_1, \lambda x_2 \in \mathbb{R}$, it follows that we have

$$\lambda(x_1, 0) = (\lambda x_1, 0) \in U_1,$$
$$\lambda(x_1, x_1) = (\lambda x_1, \lambda x_1) \in U_2,$$
$$\lambda(0, x_2) = (0, \lambda x_2) \in W.$$

  Therefore, U_1, U_2, W are all closed under scalar multiplication.

Since we satisfied all the properties of a subspace, we conclude that U_1, U_2, W are subspaces of $\mathbb{C}^{\mathbb{R}}$. Next, we will show that U_1, U_2, W satisfy $U_1 + W = U_2 + W$. Using the definition of the sum of subsets, we have

$$\begin{aligned} U_1 + W &= \{(x_1, 0) + (0, x_2) : (x_1, 0) \in U_1, (0, x_2) \in U_2, x_1, x_2 \in \mathbb{R}\} \\ &= \{(x_1, x_2) : x_1, x_2 \in \mathbb{R}\} \\ &= \mathbb{R}^2 \end{aligned}$$

and

$$\begin{aligned} U_2 + W &= \{(x_1, x_1) + (0, x_2) : (x_1, x_1) \in U_2, (0, x_2) \in U_2, x_1, x_2 \in \mathbb{R}\} \\ &= \{(x_1, x_1 + x_2) : x_1, x_2 \in \mathbb{R}\} \\ &= \mathbb{R}^2, \end{aligned}$$

and so we conclude $U_1 + W = \mathbb{R}^2 = U_2 + W$. Finally, we will show $U_1 \neq U_2$. Consider the vector $(1, 0) \in \mathbb{R}^2$. Then we have $(1, 0) \in U_1$ and $(1, 0) \notin U_2$. As we found an element that belongs to U_1 but not U_2, we conclude $U_1 \neq U_2$. □

1.C.24. Let $\mathbb{R}^{\mathbb{R}}$ be the set of all real-valued functions $f : \mathbb{R} \to \mathbb{R}$. A real-valued function $f : \mathbb{R} \to \mathbb{R}$ is called *even* if

$$f(-x) = f(x)$$

for all $x \in \mathbb{R}$. A real-valued function $f : \mathbb{R} \to \mathbb{R}$ is called *odd* if

$$f(-x) = -f(x)$$

for all $x \in \mathbb{R}$. Let U_e denote the set of real-valued even functions on $\mathbb{R}$, and let U_o denote the set of real-valued odd functions on $\mathbb{R}$. Show that we have $\mathbb{R}^{\mathbb{R}} = U_e \oplus U_o$.

Proof. First, we need to show that U_e and U_o are subspaces of $\mathbb{R}^{\mathbb{R}}$.

- Additive identity: For all $x \in \mathbb{R}$, the zero function satisfies $0(x) = 0 = 0(-x)$ and $0(x) = 0 = -0 = -0(x)$. So we have $0 \in U_e$ and $0 \in U_o$.
- Closed under addition: Let $g_1, g_2 \in U_e$ and $h_1, h_2 \in U_o$ be arbitrary. Then, for all $x \in \mathbb{R}$, we have $g_1(x) = g_1(-x)$, $g_2(x) = g_2(-x)$, $h_1(-x) = -h_1(x)$, and $h_2(-x) = -h_2(x)$. So, for all $x \in \mathbb{R}$, we have

$$\begin{aligned}(g_1 + g_2)(-x) &= g_1(-x) + g_2(-x)\\ &= g_1(x) + g_2(x)\\ &= (g_1 + g_2)(x)\end{aligned}$$

  and

$$\begin{aligned}(h_1 + h_2)(-x) &= h_1(-x) + h_2(-x)\\ &= -h_1(x) - h_2(x)\\ &= -(h_1(x) + h_2(x))\\ &= -(h_1 + h_2)(x).\end{aligned}$$

  So we have $g_1 + g_2 \in U_e$ and $h_1 + h_2 \in U_o$.
- Closed under scalar multiplication: Let $\lambda \in \mathbb{F}$, $g \in U_e$, and $h \in U_o$ be arbitrary. Then, for all $x \in \mathbb{R}$, we have $g(-x) = g(x)$ and $h(-x) = -h(x)$. So, for all $x \in \mathbb{R}$, we have

$$\begin{aligned}(\lambda g)(-x) &= \lambda g(-x)\\ &= \lambda g(x)\\ &= (\lambda g)(x)\end{aligned}$$

  and

$$\begin{aligned}(\lambda h)(-x) &= \lambda h(-x)\\ &= \lambda(-h(x))\\ &= -\lambda h(x)\\ &= (\lambda h)(x).\end{aligned}$$

  So we have $\lambda g \in U_e$ and $\lambda g \in U_o$.

Since we satisfied all the properties of a subspace, we conclude that U_e and U_o are subspaces of $\mathbb{R}^{\mathbb{R}}$. Next, we need to show $\mathbb{R}^{\mathbb{R}} = U_e + U_o$. In other words, we will show that we can write every function $f \in \mathbb{R}^{\mathbb{R}}$ as a sum of an even function and an odd function. Define for all $x \in \mathbb{R}$ the functions $g, h \in \mathbb{R}^{\mathbb{R}}$ by

$$g(x) = \frac{f(x) + f(-x)}{2}$$

and

$$h(x) = \frac{f(x) - f(-x)}{2}.$$

Then

$$\begin{aligned}g(-x) &= \frac{f(-x) + f(-(-x))}{2}\\ &= \frac{f(-x) + f(x)}{2}\\ &= \frac{f(x) + f(-x)}{2}\\ &= g(x),\end{aligned}$$

which means g is even, or $g \in U_e$. Similarly,

$$\begin{aligned} h(-x) &= \frac{f(-x) - f(-(-x))}{2} \\ &= -\frac{f(x) - f(-x)}{2} \\ &= -h(x), \end{aligned}$$

which means h is odd, or $h \in U_o$. Finally, for all $x \in \mathbb{R}$, we have

$$\begin{aligned} (g+h)(x) &= g(x) + h(x) \\ &= \frac{f(x) + f(-x)}{2} + \frac{f(x) - f(-x)}{2} \\ &= \frac{(f(x) + f(-x)) + (f(x) - f(-x))}{2} \\ &= \frac{2f(x)}{2} \\ &= f(x), \end{aligned}$$

and so $f = g + h$, which establishes $\mathbb{R}^{\mathbb{R}} = U_e + U_o$. At this point, it remains to show $\mathbb{R}^{\mathbb{R}} = U_e \oplus U_o$. According to 1.45 of Axler, we only need to show $U_e \cap U_o = \{0\}$. So suppose we have $f \in U_e \cap U_o$. Then $f \in U_e$ and $f \in U_o$, which means f is both even and odd. In other words, f satisfies both $f(-x) = f(x)$ and $f(-x) = -f(x)$ for all $x \in \mathbb{R}$. Combining the two equations gives us $-f(x) = f(x)$, which implies $f(x) = 0$ for all $x \in \mathbb{R}$. Therefore, $f = 0 \in \{0\}$, and so we have $U_e \cap U_0 \subset \{0\}$. On the other hand, since $U_e \cap U_o$ is a subspace of $\mathbb{R}^{\mathbb{R}}$, we have in fact the set equality $U_e \cap U_0 = \{0\}$. By 1.45 of Axler, we conclude $\mathbb{R}^{\mathbb{R}} = U_e \oplus U_o$. □