MATH 131: Linear Algebra I
University of California, Riverside
Homework 2 Solutions
July 8, 2019

Solutions to assigned homework problems from *Linear Algebra Done Right* (third edition) by Sheldon Axler

2.A: 1, 3, 5, 6, 7, 8, 9, 10, 11
2.B: 3, 4, 6, 8
2.C: 1, 2, 3, 11, 12, 13, 15, 16, 17

2.A.1. Let V be a vector space. Suppose v_1, v_2, v_3, v_4 is a basis of V. Prove that the list

$$v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4$$

is also a basis of V.

Proof. Suppose $a_1, a_2, a_3, a_4 \in \mathbb{F}$ satisfy

$$a_1(v_1 - v_2) + a_2(v_2 - v_3) + a_3(v_3 - v_4) + a_4 v_4 = 0.$$

Algebraically rearranging the terms, we get

$$a_1 v_1 + (-a_1 + a_2)v_2 + (-a_2 + a_3)v_3 + (-a_3 + a_4)v_4 = 0.$$

Since v_1, v_2, v_3, v_4 is linearly independent, all scalars are zero, which means we have

$$a_1 = 0, -a_1 + a_2 = 0, -a_2 + a_3 = 0, -a_3 + a_4 = 0.$$

The second equation $-a_1 + a_2 = 0$ with $a_1 = 0$ implies $a_2 = 0$. The third equation $-a_2 + a_3 = 0$ with $a_2 = 0$ implies $a_3 = 0$. The fourth equation $-a_3 + a_4 = 0$ with $a_3 = 0$ implies $a_4 = 0$. So we have

$$a_1 = 0, a_2 = 0, a_3 = 0, a_4 = 0,$$

and so we conclude $v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4$ is linearly independent. Next, we need to prove that $v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4$ spans V. Since v_1, v_2, v_3, v_4 spans V, there exist $a_1, a_2, a_3, a_4 \in \mathbb{F}$ such that

$$v = a_1 v_1 + a_2 v_2 + a_3 v_3 + a_4 v_4.$$

Furthermore, observe that we can write

$$\begin{aligned} v_1 &= (v_1 - v_2) + (v_2 - v_3) + (v_3 - v_4) + v_4, \\ v_2 &= (v_2 - v_3) + (v_3 - v_4) + v_4, \\ v_3 &= (v_3 - v_4) + v_4. \end{aligned}$$

So we have

$$\begin{aligned} v &= a_1 v_1 + a_2 v_2 + a_3 v_3 + a_4 v_4 \\ &= a_1((v_1 - v_2) + (v_2 - v_3) + (v_3 - v_4) + v_4) + a_2((v_2 - v_3) + (v_3 - v_4) + v_4) + a_3((v_3 - v_4) + v_4) + a_4 v_4 \\ &= a_1(v_1 - v_2) + (a_1 + a_2)(v_2 - v_3) + (a_1 + a_2 + a_3)(v_3 - v_4) + (a_1 + a_2 + a_3 + a_4)v_4. \end{aligned}$$

Since we also have $a_1, a_1 + a_2, a_1 + a_2 + a_3, a_1 + a_2 + a_3 + a_4 \in \mathbb{F}$, it follows that the list $v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4$ spans V. Therefore, $v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4$ is a basis of V. □

2.A.3. Find a number t such that

$$(3, 1, 4), (2, -3, 5), (5, 9, t)$$

is not linearly independent in $\mathbb{R}^3$.

Proof. To find t for which the list $(3, 1, 4), (2, -3, 5), (5, 9, t)$ is linearly independent, we can write the last vector $(5, 9, t)$ as a linear combination of the first two vectors $(3, 1, 4), (2, -3, 5)$ as follows:

$$(5, 9, t) = a_1(3, 1, 4) + a_2(2, -3, 5)$$

for some $a_1, a_2 \in \mathbb{F}$ and solve for t. To do this, we can rewrite the above equation as

$$(5, 9, t) = (3a_1 + 2a_2, a_1 - 3a_2, 4a_1 + 5a_2),$$

from which we can equate the coordinates of both sides to obtain the system of equations

$$\begin{aligned} 5 &= 3a_1 + 2a_2, \\ 9 &= a_1 - 3a_2, \\ t &= 4a_1 + 5a_2. \end{aligned}$$

The first two of the three equations in the above system can be system-solved to get $a_1 = 3, a_2 = -2$. Substituting these values into the third equation, we get $t = 4(3) + 5(2) = 22$. □

2.A.5. (a) Show that if we think of $\mathbb{C}$ as a vector space over $\mathbb{R}$, then the list $(1+i, 1-i)$ is linearly independent.

Proof. If $\mathbb{C}$ as a vector space over $\mathbb{R}$, then the scalars are real numbers. Suppose $a_1, a_2 \in \mathbb{R}$ satisfy

$$a_1(1+i) + a_2(1-i) = 0 + 0i.$$

Then using the definition of addition on $\mathbb{C}$ for the left-hand side of the above equation, we get

$$(a_1 + a_2) + (a_1 - a_2)i = 0 + 0i,$$

from which we can equate the terms from both sides to obtain the system of equations

$$\begin{aligned} a_1 + a_2 &= 0, \\ a_1 - a_2 &= 0. \end{aligned}$$

The only pair of solutions in $\mathbb{R}$ to this system of equations is $a_1 = 0, a_2 = 0$. Therefore, the list $(1+i, 1-i)$ is linearly independent. □

(b) Show that if we think of $\mathbb{C}$ as a vector spae over $\mathbb{C}$, then the list $(1+i, 1-i)$ is linearly dependent.

Proof. If $\mathbb{C}$ as a vector space over $\mathbb{C}$, then the scalars are complex numbers. Choose $c_1 = i, c_2 = 1 \in \mathbb{C}$. Then c_1, c_2 satisfy

$$\begin{aligned} c_1(1+i) + c_2(1-i) &= i(1+i) + 1(1-i) \\ &= (i + i^2) + (1-i) \\ &= (i-1) + (1-i) \\ &= 0. \end{aligned}$$

Since $c_1 = 1, c_2 = 1$ are nonzero scalars, we conclude that the list $(1+i, 1-i)$ is linearly dependent. □

2.A.6. Suppose v_1, v_2, v_3, v_4 is a linearly independent in V. Prove that the list

$$v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4$$

is also linearly independent.

Proof. Suppose $a_1, a_2, a_3, a_4 \in \mathbb{F}$ satisfy

$$a_1(v_1 - v_2) + a_2(v_2 - v_3) + a_3(v_3 - v_4) + a_4 v_4 = 0.$$

Algebraically rearranging the terms, we get

$$a_1 v_1 + (-a_1 + a_2)v_2 + (-a_2 + a_3)v_3 + (-a_3 + a_4)v_4 = 0.$$

Since v_1, v_2, v_3, v_4 is linearly independent, all scalars are zero, which means we have

$$a_1 = 0, -a_1 + a_2 = 0, -a_2 + a_3 = 0, -a_3 + a_4 = 0.$$

The second equation $-a_1 + a_2 = 0$ with $a_1 = 0$ implies $a_2 = 0$. The third equation $-a_2 + a_3 = 0$ with $a_2 = 0$ implies $a_3 = 0$. The fourth equation $-a_3 + a_4 = 0$ with $a_3 = 0$ implies $a_4 = 0$. So we have

$$a_1 = 0, a_2 = 0, a_3 = 0, a_4 = 0,$$

and so we conclude $v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4$ is linearly independent. □

2.A.7. Prove or give a counterexample: If $v_1, v_2, v_3, \ldots, v_m$ is a linearly independent list of vectors in V, then

$$5v_1 - 4v_2, v_2, v_3, \ldots, v_m$$

is linearly independent.

Proof. Suppose $a_1, \ldots, a_m \in \mathbb{F}$ satisfy

$$a_1(5v_1 - 4v_2) + a_2 v_2 + a_3 v_3 + \cdots + a_m v_m = 0.$$

Algebraically rearranging the left-hand side of the above equation gives

$$(5a_1)v_1 + (-4a_1 + a_2)v_2 + a_3 v_3 + \cdots + a_m v_m = 0.$$

Since $v_1, v_2, v_3, \ldots, v_m$ is linearly independent, all scalars are zero, which means we have

$$5a_1 = 0, -4a_1 + a_2 = 0, a_3 = 0, \ldots, a_m = 0.$$

The first equation $5a_1 = 0$ implies $a_1 = 0$. The second equation $-4a_1 + a_2 = 0$ with $a_1 = 0$ implies $a_2 = 0$. So we have

$$a_1 = 0, a_2 = 0, a_3 = 0, \ldots, a_m = 0,$$

and so we conclude that $5v_1 - 4v_2, v_2, v_3, \ldots, v_m$ is linearly independent. □

2.A.8. Prove or give a counterexample: If $v_1, v_2, v \ldots, v_m$ is a linearly independent list of vectors in V and $\lambda \in \mathbb{F}$ with $\lambda \neq 0$, then $\lambda v_1, \lambda v_2, \lambda v_3, \ldots, \lambda v_m$ is linearly independent.

Proof. Suppose $a_1, \ldots, a_m \in \mathbb{F}$ satisfy

$$a_1(\lambda v_1) + \cdots + a_m(\lambda v_m) = 0.$$

Rewriting the parentheses on the left-hand side of the above equation gives

$$(a_1\lambda)v_1 + \cdots + (a_m\lambda)v_m = 0.$$

Since $v_1, \ldots, v_m$ is linearly independent, all scalars are zero, which means we have

$$a_1\lambda = 0, \ldots, a_m\lambda = 0.$$

Because we assumed $\lambda \neq 0$, we arrive at

$$a_1 = 0, \ldots, a_m = 0,$$

and so we conclude that $\lambda v_1, \lambda v_2, \lambda v_3, \ldots, \lambda v_m$ is linearly independent. □

2.A.9. Prove or give a counterexample: If $v_1, \ldots, v_m$ and $w_1, \ldots, w_m$ are also linearly independent lists of vectors in V, then $v_1 + w_1, \ldots, v_m + w_m$ is linearly independent.

Proof. We will give a counterexample to show that this statement is false. Let $m = 2$, let $V = \mathbb{R}^2$, let $v_1 = (1, 0), v_2 = (0, 1)$ be a list of vectors in $\mathbb{R}^2$, and let $w_1 = -v_1 = (-1, 0)$ and $w_2 = -v_2 = (0, -1)$. Suppose a_1, a_2 satisfy

$$a_1v_1 + a_2v_2 = (0, 0).$$

Then we have

$$\begin{aligned}(0, 0) &= a_1v_1 + a_2v_2 \\ &= a_1(1, 0) + a_2(0, 1) \\ &= (a_1, 0) + (0, a_2) \\ &= (a_1, a_2),\end{aligned}$$

from which we get $a_1 = 0, a_2 = 0$, and so v_1, v_2 is linearly independent. Similarly, we have

$$\begin{aligned}(0, 0) &= b_1w_1 + b_2w_2 \\ &= b_1(-1, 0) + b_2(0, -1) \\ &= (-b_1, 0) + (0, -b_2) \\ &= (-b_1, -b_2),\end{aligned}$$

from which we get $-b_1 = 0, -b_2 = 0$, or equivalently $b_1 = 0, b_2 = 0$, and so w_1, w_2 is linearly independent. However, if we choose $c_1 = 1, c_2 = 1$, then we have

$$\begin{aligned}c_1(v_1 + w_1) + c_2(v_2 + w_2) &= 1((1, 0) + (-1, 0)) + 1((0, 1) + (0, -1)) \\ &= 1(0, 0) + 1(0, 0) \\ &= (0, 0) + (0, 0) \\ &= (0, 0),\end{aligned}$$

which means $v_1 + w_1, v_2 + w_2$ is not linearly independent. □

2.A.10. Suppose $v_1, \ldots, v_m$ is linearly dependent in V and $w \in V$. Prove that if $v_1 + w, \ldots, v_m + w$ is linearly dependent, then $w \in \operatorname{span}(v_1, \ldots, v_m)$.

Proof. Suppose $v_1 + w, \ldots, v_m + w$ is linearly independent. Then there exist scalars $a_1, \ldots, a_m$, not all zero, that satisfy

$$a_1(v_1 + w) + \cdots + a_m(v_m + w) = 0.$$

We can algebraically rearrange the terms in the left-hand side of the above equation to get

$$a_1v_1 + \cdots + a_mv_m + (a_1 + \cdots + a_m)w = 0.$$

We claim that, if $a_1, \ldots, a_m$ are not all zero, then we have $a_1 + \cdots + a_m \neq 0$. To prove this claim, suppose by contradiction that we have $a_1 + \cdots + a_m = 0$. Then the last equation reduces to

$$a_1v_1 + \cdots + a_mv_m = 0.$$

According to the premises, $v_1, \dots, v_m$ is linearly independent in V, which means all the scalars are zero:

$$a_1 = 0, \dots, a_m = 0,$$

which contradicts our earlier result saying that not all the scalars $a_1, \dots, a_m$ are zero. Therefore, we proved our claim, and so we have $a_1 + \cdots + a_m \neq 0$. Therefore, we can use the above equation

$$a_1 v_1 + \cdots + a_m v_m + (a_1 + \cdots + a_m)w = 0.$$

to obtain

$$w = \left(-\frac{a_1}{a_1 + \cdots + a_m}\right) v_1 + \cdots + \left(-\frac{a_m}{a_1 + \cdots + a_m}\right) v_m.$$

Since we have $-\frac{a_1}{a_1+\cdots+a_m}, \dots, -\frac{a_m}{a_1+\cdots+a_m} \in \mathbb{F}$, we conclude w is a linear combination of the vectors $v_1, \dots, v_m$, and so we have $w \in \operatorname{span}(v_1, \dots, v_m)$. □

2.A.11. Suppose $v_1, \dots, v_m$ is linearly independent in V and $w \in V$. Show that $v_1, \dots, v_m, w$ is linearly independent if and only if

$$w \notin \operatorname{span}(v_1, \dots, v_m).$$

Proof. Forward direction: If $v_1, \dots, v_m, w$ is linearly independent, then $w \notin \operatorname{span}(v_1, \dots, v_m)$. Suppose $v_1, \dots, v_m, w$ is linearly independent, and suppose by contradiction that we have $w \in \operatorname{span}(v_1, \dots, v_m)$. Then there exist $a_1, \dots, a_m \in \mathbb{F}$ that satisfy

$$w = a_1 v_1 + \cdots + a_m v_m.$$

We can rewrite the above equation as

$$a_1 v_1 + \cdots + a_m v_m - w = 0.$$

This means that $v_1, \dots, v_m, w$ is linearly dependent, contradicting our assumption that $v_1, \dots, v_m, w$ is linearly independent.

Backward direction: If $w \notin \operatorname{span}(v_1, \dots, v_m)$, then $v_1, \dots, v_m, w$ is linearly independent. Suppose we have $w \notin \operatorname{span}(v_1, \dots, v_m)$, and suppose to the contrary that $v_1, \dots, v_m, w$ is linearly dependent. Then there exist $a_1, \dots, a_m, b \in \mathbb{F}$, not all zero, such that

$$a_1 v_1 + \cdots + a_m v_m + bw = 0.$$

At this point, we will continue our argument by breaking down into separate cases: $b = 0$ and $b \neq 0$.

- Case 1: Suppose $b = 0$. Then the equation

  $$a_1 v_1 + \cdots + a_m v_m + bw = 0$$

  reduces to

  $$a_1 v_1 + \cdots + a_m v_m = 0.$$

  Since we assumed in the premises that $v_1, \dots, v_m$ is linearly independent in V, all the scalars are zero:

  $$a_1 = 0, \dots, a_m = 0.$$

  Combining our assumption and results, we have

  $$a_1 = 0, \dots, a_m = 0, b = 0,$$

  which means $v_1, \dots, v_m, w$ is linearly independent, which contradicts our assumption that $v_1, \dots, v_m, w$ is linearly dependent.

- Case 2: Suppose $b \neq 0$. Then we can solve the equation

  $$a_1 v_1 + \cdots + a_m v_m + bw = 0$$

  to get

  $$w = \left(-\frac{a_1}{b}\right) v_1 + \cdots + \left(-\frac{a_m}{b}\right) v_m.$$

  Since we have $-\frac{a_1}{b}, \dots, -\frac{a_m}{b} \in \mathbb{F}$, we conclude that w is a linear combination of the vectors $v_1, \dots, v_m$, and so we have $w \in \operatorname{span}(v_1, \dots, v_n)$. But this contradicts our assumption $w \notin \operatorname{span}(v_1, \dots, v_n)$.

Therefore, in either case of $b = 0$ or $b \neq 0$, we achieve a contradiction, and so we conclude that $v_1, \dots, v_m, w$ is linearly independent. □

2.B.3. (a) Let U be the subspace of $\mathbb{R}^5$ defined by

$$U = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 : x_1 = 3x_2 \text{ and } x_3 = 7x_4\}.$$

Find a basis of U.

Proof. Let $(x_1, x_2, x_3, x_4, x_5) \in U$ be arbitrary. Then we have $x_1 = 3x_2$ and $x_3 = 7x_4$, and so we can write

$$\begin{aligned}(x_1, x_2, x_3, x_4, x_5) &= (3x_2, x_2, 7x_4, x_4, x_5)\\ &= (3x_2, x_2, 0, 0, 0) + (0, 0, 7x_4, x_4, 0) + (0, 0, 0, 0, x_5)\\ &= x_2(3, 1, 0, 0, 0) + x_4(0, 0, 7, 1, 0) + x_5(0, 0, 0, 0, 1).\end{aligned}$$

Since we have $x_2, x_4, x_5 \in \mathbb{R}$, we have established that the list $(3, 1, 0, 0, 0), (0, 0, 7, 1, 0), (0, 0, 0, 0, 1)$ spans U. If we can also show that the list is also linearly independent in U, then it would in fact be a basis of U. Suppose $a_1, a_2, a_3 \in \mathbb{R}$ satisfy

$$a_1(3, 1, 0, 0, 0) + a_2(0, 0, 7, 1, 0) + a_3(0, 0, 0, 0, 1) = (0, 0, 0, 0, 0).$$

Applying addition and scalar multiplication in $\mathbb{R}^5$ to the left-hand side of the above equation, we get

$$(3a_1, a_1, 7a_2, a_2, a_3) = (0, 0, 0, 0, 0),$$

from which we can equate the second, fourth, and fifth coordinates of both sides to obtain

$$a_1 = 0, a_2 = 0, a_3 = 0,$$

and so the list $(3, 1, 0, 0, 0), (0, 0, 7, 1, 0), (0, 0, 0, 0, 1)$ is linearly independent in U. So this list is a basis of U. □

(b) Extend the basis in part (a) to a basis of $\mathbb{R}^5$.

Proof. Adjoin the vectors $(1, 0, 0, 0, 0), (0, 1, 0, 0, 0)$ to the basis $(3, 1, 0, 0, 0), (0, 0, 7, 1, 0), (0, 0, 0, 0, 1)$ of U in order to form the list $(3, 1, 0, 0, 0), (0, 0, 7, 1, 0), (0, 0, 0, 0, 1), (0, 1, 0, 0, 0), (0, 0, 1, 0, 0)$ in $\mathbb{R}^5$. We need to show that this resulting list is in fact a basis of $\mathbb{R}^5$. We need to show that this list is linearly independent. Suppose $a_1, a_2, a_3, a_4, a_5 \in \mathbb{R}^5$ satisfy

$$a_1(3, 1, 0, 0, 0) + a_2(0, 0, 7, 1, 0) + a_3(0, 0, 0, 0, 1) + a_4(0, 1, 0, 0, 0) + a_5(0, 0, 1, 0, 0) = (0, 0, 0, 0, 0).$$

Applying addition and scalar multiplication in $\mathbb{R}^5$ to the left-hand side of the above equation, we get

$$(3a_1, a_1 + a_4, 7a_2 + a_5, a_2, a_3) = (0, 0, 0, 0, 0),$$

from which we can equate the coordinates of both sides to obtain

$$3a_1 = 0, a_1 + a_4 = 0, 7a_2 + a_5 = 0, a_2 = 0, a_3 = 0.$$

The first equation $3a_1 = 0$ implies $a_1 = 0$, the second equation with $a_1 = 0$ implies $a_4 = 0$, and the third equation $7a_2 + a_5 = 0$ with $a_2 = 0$ implies $a_5 = 0$. Therefore, we have

$$a_1 = 0, a_2 = 0, a_3 = 0, a_4 = 0, a_5 = 0,$$

and so the list $(3, 1, 0, 0, 0), (0, 0, 7, 1, 0), (0, 0, 0, 0, 1), (0, 1, 0, 0, 0), (0, 0, 1, 0, 0)$ is linearly independent in $\mathbb{R}^5$. Furthermore, since $(3, 1, 0, 0, 0), (0, 0, 7, 1, 0), (0, 0, 0, 0, 1), (0, 1, 0, 0, 0), (0, 0, 1, 0, 0)$ has length 5 and we have $\dim \mathbb{R}^5 = 5$, it is of the right length, which means, by 2.39 of Axler, this list is a basis of $\mathbb{R}^5$. □

(c) Find a subspace W of $\mathbb{R}^5$ such that $\mathbb{R}^5 = U \oplus W$.

Proof. Following the proof of 2.34 of Axler, let $W = \operatorname{span}((0, 1, 0, 0, 0), (0, 0, 1, 0, 0))$. Then, by 2.7 of Axler, W is a subspace of $\mathbb{R}^5$. To prove $\mathbb{R}^5 = U \oplus W$, we need to show $\mathbb{R}^5 = U + W$ and $U \cap W = \{(0, 0, 0, 0, 0)\}$, according to 1.45 of Axler. To prove $\mathbb{R}^5 = U + W$, let $(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5$ be a vector. We need to show that $(x_1, x_2, x_3, x_4, x_5)$ is a sum of a vector in U and a vector in W. By part (b), we have the basis $(3, 1, 0, 0, 0), (0, 0, 7, 1, 0), (0, 0, 0, 0, 1), (0, 1, 0, 0, 0), (0, 0, 1, 0, 0)$ of V, which means it is a list that spans V. So there exist $a_1, a_2, a_3, b_1, b_2 \in \mathbb{R}$ that satisfy

$$(x_1, x_2, x_3, x_4, x_5) = u + w,$$

where

$$u = a_1(3, 1, 0, 0, 0) + a_2(0, 0, 7, 1, 0) + a_3(0, 0, 0, 0, 1)$$

and

$$w = b_1(0, 1, 0, 0, 0) + b_2(0, 0, 1, 0, 0).$$

Since we have $(3, 1, 0, 0, 0), (0, 0, 7, 1, 0), (0, 0, 0, 0, 1) \in U$ and, as subspaces, U and W are closed under addition in V, we have $u = a_1(3, 1, 0, 0, 0) + a_2(0, 0, 7, 1, 0) + a_3(0, 0, 0, 0, 1) \in U$ and $w = b_1(0, 1, 0, 0, 0) + b_2(0, 0, 1, 0, 0) \in W$. So we conclude $(x_1, x_2, x_3, x_4, x_5) \in U + W$, and so $\mathbb{R}^5 \subset U + W$. However, according to 1.39 of Axler, $U + W$ is a subspace of $\mathbb{R}^5$. So we must have the set equality $\mathbb{R}^5 = U + W$. We are now left to prove $U \cap W = \{(0, 0, 0, 0, 0)\}$. Suppose we have $(x_1, x_2, x_3, x_4, x_5) \in U \cap W$. Then we have $(x_1, x_2, x_3, x_4, x_5) \in U$ and $(x_1, x_2, x_3, x_4, x_5) \in W$. According to our proof of part (a), the list $(3, 1, 0, 0, 0), (0, 0, 7, 1, 0), (0, 0, 0, 0, 1)$ is a basis of U, and so it spans U. Since we originally let $W = \operatorname{span}((0, 1, 0, 0, 0), (0, 0, 1, 0, 0))$, by this construction the list $(0, 1, 0, 0, 0), (0, 0, 1, 0, 0)$ spans W. So

$(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5$ can be written as a linear combination of the vectors in the two lists. In other words, there exist scalars $a_1, a_2, a_3, b_1, b_2 \in \mathbb{R}$ that saitsfy

$$(x_1, x_2, x_3, x_4, x_5) = a_1(3, 1, 0, 0, 0) + a_2(0, 0, 7, 1, 0) + a_3(0, 0, 0, 0, 1)$$

and

$$(x_1, x_2, x_3, x_4, x_5) = b_1(0, 1, 0, 0, 0) + b_2(0, 0, 1, 0, 0).$$

Equating the two equations, we get

$$a_1(3, 1, 0, 0, 0) + a_2(0, 0, 7, 1, 0) + a_3(0, 0, 0, 0, 1) - b_1(0, 1, 0, 0, 0) - b_2(0, 0, 1, 0, 0) = (0, 0, 0, 0, 0).$$

Since, according to our proof of part (b), the list $(3, 1, 0, 0, 0), (0, 0, 7, 1, 0), (0, 0, 0, 0, 1), (0, 1, 0, 0, 0), (0, 0, 1, 0, 0)$ is a basis of $\mathbb{R}^5$, it is linearly independent in $\mathbb{R}^5$. So all the scalars are zero; that is, we have

$$a_1 = 0, a_2 = 0, a_3 = 0, b_1 = 0, b_2 = 0.$$

Therefore, we have

$$\begin{aligned}(x_1, x_2, x_3, x_4, x_5) &= u + w \\ &= (a_1(3, 1, 0, 0, 0) + a_2(0, 0, 7, 1, 0) + a_3(0, 0, 0, 0, 1)) + (b_1(0, 1, 0, 0, 0) + b_2(0, 0, 1, 0, 0)) \\ &= (0(3, 1, 0, 0, 0) + 0(0, 0, 7, 1, 0) + 0(0, 0, 0, 0, 1)) + (0(0, 1, 0, 0, 0) + 0(0, 0, 1, 0, 0)) \\ &= ((0, 0, 0, 0, 0) + (0, 0, 0, 0, 0) + (0, 0, 0, 0, 0)) + ((0, 0, 0, 0, 0) + (0, 0, 0, 0, 0)) \\ &= (0, 0, 0, 0, 0).\end{aligned}$$

Therefore, we have $U \cap W \subset \{(0, 0, 0, 0, 0)\}$. As subspaces, U and W contain the zero vector; that is, we have $(0, 0, 0, 0, 0) \in U$ and $(0, 0, 0, 0, 0) \in W$. So we have $(0, 0, 0, 0, 0) \in U \cap W$, and so $\{(0, 0, 0, 0, 0)\} \subset U \cap W$. Therefore, we obtain the set equality $U \cap W = \{(0, 0, 0, 0, 0)\}$. So we established $\mathbb{R}^5 = V + W$ and $V \cap W = \{(0, 0, 0, 0, 0)\}$. By 1.45 of Axler, we conclude $\mathbb{R}^5 = U \oplus W$. □

2.B.4. (a) Let U be the subspace of $\mathbb{C}^5$ defined by

$$U = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{C}^5 : 6z_1 = z_2 \text{ and } z_3 + 2z_4 + 3z_5 = 0\}.$$

Find a basis of U.

Proof. Let $(z_1, z_2, z_3, z_4, z_5) \in U$ be arbitrary. Then we have $6z_1 = z_2$ and $z_3 + 2z_4 + z_5 = 0$, and so we can write

$$\begin{aligned}(z_1, z_2, z_3, z_4, z_5) &= (z_1, 6z_1, -2z_4 - z_5, z_4, z_5) \\ &= (z_1, 6z_1, 0, 0, 0) + (0, 0, -2z_4, z_4, 0) + (0, 0, 0, -z_5, z_5) \\ &= z_1(1, 6, 0, 0, 0) + z_4(0, 0, -2, 1, 0) + z_5(0, 0, 0, -1, 1).\end{aligned}$$

Since we have $z_1, z_4, z_5 \in \mathbb{R}$, we have established that the list $(1, 6, 0, 0, 0), (0, 0, -2, 1, 0), (0, 0, 0, -1, 1)$ spans U. If we can also show that the list is also linearly independent in U, then it would in fact be a basis of U. Suppose $a_1, a_2, a_3 \in \mathbb{C}$ satisfy

$$a_1(1, 6, 0, 0, 0) + a_2(0, 0, -2, 1, 0) + a_3(0, 0, 0, -1, 1) = (0, 0, 0, 0, 0).$$

Applying addition and scalar multiplication in $\mathbb{R}^5$ to the left-hand side of the above equation, we get

$$(a_1, 6a_1, -2a_2, a_2 - a_3, a_3) = (0, 0, 0, 0, 0),$$

from which we can equate the first, thid, and fifth coordinates of both sides to obtain

$$a_1 = 0, -2a_2 = 0, a_3 = 0.$$

Since $-2a_2 = 0$ implies $a_2 = 0$, we get

$$a_1 = 0, a_2 = 0, a_3 = 0,$$

and so the list $(1, 6, 0, 0, 0), (0, 0, -2, 1, 0), (0, 0, 0, -1, 1)$ is linearly independent in U. So this list is a basis of U. □

(b) Extend the basis in part (a) to a basis of $\mathbb{C}^5$.

Proof. Adjoin the vectors $(1, 0, 0, 0, 0), (0, 1, 0, 0, 0)$ to the basis $(1, 6, 0, 0, 0), (0, 0, -2, 1, 0), (0, 0, 0, -1, 1)$ of U in order to form the list $(1, 6, 0, 0, 0), (0, 0, -2, 1, 0), (0, 0, 0, -1, 1), (0, 1, 0, 0, 0), (0, 0, 1, 0, 0)$ in $\mathbb{C}^5$. We need to show that this resulting list is in fact a basis of $\mathbb{C}^5$. We need to show that this list is linearly independent. Suppose $a_1, a_2, a_3, a_4, a_5 \in \mathbb{C}^5$ satisfy

$$a_1(1, 6, 0, 0, 0) + a_2(0, 0, -2, 1, 0) + a_3(0, 0, 0, -1, 1) + a_4(0, 1, 0, 0, 0) + a_5(0, 0, 1, 0, 0) = (0, 0, 0, 0, 0).$$

Applying addition and scalar multiplication in $\mathbb{C}^5$ to the left-hand side of the above equation, we get

$$(a_1, 6a_1 + a_4, -2a_3 + a_5, a_2 - a_3, a_3) = (0, 0, 0, 0, 0),$$

from which we can equate the coordinates of both sides to obtain

$$a_1 = 0, 6a_1 + a_4 = 0, -2a_3 + a_5 = 0, a_2 - a_3 = 0, a_3 = 0.$$

The second equation $6a_1 + a_4 = 0$ with $a_1 = 0$ implies $a_4 = 0$, the third equation with $a_3 = 0$ implies $a_5 = 0$, and the fourth equation $a_2 - a_3 = 0$ with $a_3 = 0$ implies $a_2 = 0$. Therefore, we have

$$a_1 = 0, a_2 = 0, a_3 = 0, a_4 = 0, a_5 = 0,$$

and so the list $(1, 6, 0, 0, 0), (0, 0, -2, 1, 0), (0, 0, 0, -1, 1), (0, 1, 0, 0, 0), (0, 0, 1, 0, 0)$ is linearly independent in $\mathbb{C}^5$. Furthermore, since $(1, 6, 0, 0, 0), (0, 0, -2, 1, 0), (0, 0, 0, -1, 1), (0, 1, 0, 0, 0), (0, 0, 1, 0, 0)$ has length 5 and we have $\dim \mathbb{C}^5 = 5$, it is of the right length, which means, by 2.39 of Axler, this list is a basis of $\mathbb{C}^5$. □

(c) Find a subspace W of $\mathbb{C}^5$ such that $\mathbb{C}^5 = U \oplus W$.

Proof. Following the proof of 2.34 of Axler, let $W = \operatorname{span}((0, 1, 0, 0, 0), (0, 0, 1, 0, 0))$. Then, by 2.7 of Axler, W is a subspace of $\mathbb{C}^5$. To prove $\mathbb{C}^5 = U \oplus W$, we need to show $\mathbb{C}^5 = U + W$ and $U \cap W = \{(0, 0, 0, 0, 0)\}$, according to 1.45 of Axler. To prove $\mathbb{C}^5 = U + W$, let $(z_1, z_2, z_3, z_4, z_5) \in \mathbb{R}^5$ be a vector. We need to show that $(z_1, z_2, z_3, z_4, z_5)$ is a sum of a vector in U and a vector in W. By part (b), we have the basis $(1, 6, 0, 0, 0), (0, 0, -2, 1, 0), (0, 0, 0, -1, 1), (0, 1, 0, 0, 0), (0, 0, 1, 0, 0)$ of V, which means it is a list that spans V. So there exist $a_1, a_2, a_3, b_1, b_2 \in \mathbb{C}$ that satisfy

$$(z_1, z_2, z_3, z_4, z_5) = u + w,$$

where

$$u = a_1(1, 6, 0, 0, 0) + a_2(0, 0, -2, 1, 0) + a_3(0, 0, 0, -1, 1)$$

and

$$w = b_1(0, 1, 0, 0, 0) + b_2(0, 0, 1, 0, 0).$$

Since we have $(1, 6, 0, 0, 0), (0, 0, -2, 1, 0), (0, 0, 0, -1, 1) \in U$ and, as subspaces, U and W are closed under addition in V, we have $u = a_1(1, 6, 0, 0, 0) + a_2(0, 0, -2, 1, 0) + a_3(0, 0, 0, -1, 1) \in U$ and $w = b_1(0, 1, 0, 0, 0) + b_2(0, 0, 1, 0, 0) \in W$. So we conclude $(z_1, z_2, z_3, z_4, z_5) \in U + W$, and so $\mathbb{C}^5 \subset U + W$. However, according to 1.39 of Axler, $U + W$ is a subspace of $\mathbb{R}^5$. So we must have the set equality $\mathbb{C}^5 = U + W$. We are now left to prove $U \cap W = \{(0, 0, 0, 0, 0)\}$. Suppose we have $(x_1, x_2, x_3, x_4, x_5) \in U \cap W$. Then we have $(x_1, x_2, x_3, x_4, x_5) \in U$ and $(x_1, x_2, x_3, x_4, x_5) \in W$. According to our proof of part (a), the list $(3, 1, 0, 0, 0), (0, 0, 7, 1, 0), (0, 0, 0, 0, 1)$ is a basis of U, and so it spans U. Since we originally let $W = \operatorname{span}((0, 1, 0, 0, 0), (0, 0, 1, 0, 0))$, by this construction the list $(0, 1, 0, 0, 0), (0, 0, 1, 0, 0)$ spans W. So $(z_1, z_2, z_3, z_4, z_5) \in \mathbb{C}^5$ can be written as a linear combination of the vectors in the two lists. In other words, there exist scalars $a_1, a_2, a_3, b_1, b_2 \in \mathbb{C}$ that saitsfy

$$(z_1, z_2, z_3, z_4, z_5) = a_1(1, 6, 0, 0, 0) + a_2(0, 0, -2, 1, 0) + a_3(0, 0, 0, -1, 1)$$

and

$$(z_1, z_2, z_3, z_4, z_5) = b_1(0, 1, 0, 0, 0) + b_2(0, 0, 1, 0, 0).$$

Equating the two equations, we get

$$a_1(1, 6, 0, 0, 0) + a_2(0, 0, -2, 1, 0) + a_3(0, 0, 0, -1, 1) - b_1(0, 1, 0, 0, 0) - b_2(0, 0, 1, 0, 0) = (0, 0, 0, 0, 0).$$

Since, according to our proof of part (b), the list $(1, 6, 0, 0, 0), (0, 0, -2, 1, 0), (0, 0, 0, -1, 1), (0, 1, 0, 0, 0), (0, 0, 1, 0, 0)$ is a basis of $\mathbb{C}^5$, it is linearly independent in $\mathbb{C}^5$. So all the scalars are zero; that is, we have

$$a_1 = 0, a_2 = 0, a_3 = 0, b_1 = 0, b_2 = 0.$$

Therefore, we have

$$\begin{aligned}
(z_1, z_2, z_3, z_4, z_5) &= u + w \\
&= (a_1(1, 6, 0, 0, 0) + a_2(0, 0, -2, 1, 0) + a_3(0, 0, 0, -1, 1)) + (b_1(0, 1, 0, 0, 0) + b_2(0, 0, 1, 0, 0)) \\
&= (0(1, 6, 0, 0, 0) + 0(0, 0, -2, 1, 0) + 0(0, 0, 0, -1, 1)) + (0(0, 1, 0, 0, 0) + 0(0, 0, 1, 0, 0)) \\
&= ((0, 0, 0, 0, 0) + (0, 0, 0, 0, 0) + (0, 0, 0, 0, 0)) + ((0, 0, 0, 0, 0) + (0, 0, 0, 0, 0)) \\
&= (0, 0, 0, 0, 0).
\end{aligned}$$

Therefore, we have $U \cap W \subset \{(0, 0, 0, 0, 0)\}$. As subspaces, U and W contain the zero vector; that is, we have $(0, 0, 0, 0, 0) \in U$ and $(0, 0, 0, 0, 0) \in W$. So we have $(0, 0, 0, 0, 0) \in U \cap W$, and so $\{(0, 0, 0, 0, 0)\} \subset U \cap W$. Therefore, we obtain the set equality $U \cap W = \{(0, 0, 0, 0, 0)\}$. So we established $\mathbb{C}^5 = V + W$ and $V \cap W = \{(0, 0, 0, 0, 0)\}$. By 1.45 of Axler, we conclude $\mathbb{C}^5 = U \oplus W$. □

2.B.6. Suppose v_1, v_2, v_3, v_4 is a basis of V. Prove that the list

$$v_1 + v_2, v_2 + v_3, v_3 + v_4, v_4$$

is also a basis of V.

Proof. Suppose $a_1, a_2, a_3, a_4 \in \mathbb{F}$ satisfy

$$a_1(v_1 + v_2) + a_2(v_2 + v_3) + a_3(v_3 + v_4) + a_4 v_4 = 0.$$

Algebraically rearranging the terms, we get

$$a_1 v_1 + (a_1 + a_2)v_2 + (a_2 + a_3)v_3 + (a_3 + a_4)v_4 = 0.$$

Since v_1, v_2, v_3, v_4 is linearly independent, all scalars are zero, which means we have

$$a_1 = 0, a_1 + a_2 = 0, a_2 + a_3 = 0, a_3 + a_4 = 0.$$

The second equation $a_1 + a_2 = 0$ with $a_1 = 0$ implies $a_2 = 0$. The second equation $a_2 + a_3 = 0$ with $a_2 = 0$ implies $a_3 = 0$. The third equation $a_3 + a_4 = 0$ with $a_3 = 0$ implies $a_4 = 0$. So we have

$$a_1 = 0, a_2 = 0, a_3 = 0, a_4 = 0,$$

and so we conclude $v_1 + v_2, v_2 + v_3, v_3 + v_4, v_4$ is linearly independent. Next, we need to prove that $v_1 + v_2, v_2 + v_3, v_3 + v_4, v_4$ spans V. Since v_1, v_2, v_3, v_4 spans V, there exist $a_1, a_2, a_3, a_4 \in \mathbb{F}$ such that

$$v = a_1 v_1 + a_2 v_2 + a_3 v_3 + a_4 v_4.$$

Furthermore, observe that we can write

$$\begin{aligned}
v_1 &= (v_1 + v_2) - (v_2 + v_3) - (v_3 + v_4) - v_4, \\
v_2 &= (v_2 + v_3) - (v_3 + v_4) - v_4, \\
v_3 &= (v_3 + v_4) - v_4.
\end{aligned}$$

So we have

$$\begin{aligned}
v &= a_1 v_1 + a_2 v_2 + a_3 v_3 + a_4 v_4 \\
&= a_1((v_1 + v_2) - (v_2 + v_3) - (v_3 + v_4) - v_4) + a_2((v_2 + v_3) - (v_3 + v_4) - v_4) + a_3((v_3 + v_4) - v_4) + a_4 v_4 \\
&= a_1(v_1 + v_2) + (-a_1 + a_2)(v + 2 + v_3) + (-a_1 - a_2 + a_3)v_3 + (-a_1 - a_2 - a_3 + a_4)v_4.
\end{aligned}$$

Since we also have $a_1, -a_1 + a_2, -a_1 - a_2 + a_3, -a_1 - a_2 - a_3 + a_4 \in \mathbb{F}$, it follows that the list $v_1 + v_2, v_2 + v_3, v_3 + v_4, v_4$ spans V. Therefore, $v_1 + v_2, v_2 + v_3, v_3 + v_4, v_4$ is a basis of V. □

2.B.8. Suppose U and W are subspaces of V such that $V = U \oplus W$. Suppose also that $u_1, \ldots, u_m$ is a basis of U and $w_1, \ldots, w_n$ is a basis of W. Prove that

$$u_1, \ldots, u_m, w_1, \ldots, w_n$$

is a basis of V.

Proof. First, we will show that $u_1, \ldots, u_m, w_1, \ldots, w_n$ is linearly independent. Suppose $a_1, \ldots, a_m, b_1, \ldots, b_n \in \mathbb{F}$ satisfy

$$a_1 u_1 + \cdots + a_m u_m + b_1 w_1 + \cdots + b_n w_n = 0.$$

Since U and W are subspaces of V, in particular they are closed in addition, which means we have $a_1 u_1 + \cdots + a_m u_m \in U$ and $b_1 w_1 + \cdots + b_n w_n \in W$. But the above equation $a_1 u_1 + \cdots + a_m u_m + b_1 w_1 + \cdots + b_n w_n = 0$ implies that we also have

$$a_1 u_1 + \cdots + a_m u_m = -(b_1 w_1 + \cdots + b_n w_n) \in W$$

and

$$b_1 w_1 + \cdots + b_n w_n = -(a_1 u_1 + \cdots + a_m u_m) \in U,$$

since again U and W are subspaces, which means in particular that they are closed under scalar multiplication as well. Altogether, we have

$$a_1 u_1 + \cdots + a_m u_m, b_1 w_1 + \cdots + b_n w_n \in U \cap W.$$

Since we also assumed $V = U \oplus W$, by 1.45 of Axler we have $U \cap W = \{0\}$. So we get

$$a_1 u_1 + \cdots + a_m u_m = 0$$

and

$$b_1 w_1 + \cdots + b_n w_n = 0.$$

Since $u_1, \ldots, u_m$ is a basis of U, it is linearly independent in U and spans U. In other words, $a_1 u_1 + \cdots + a_m u_m = 0$ implies

$$a_1 = 0, \ldots, a_m = 0,$$

and every vector $u \in U$ can be written
$$u = a_1 u_1 + \cdots + a_m u_m$$
for some $a_1, \ldots, a_m \in \mathbb{F}$. Similarly, since $w_1, \ldots, w_m$ is a basis of W, it is linearly independent in W and spans W. In other words, $b_1 w_1 + \cdots + b_n w_n = 0$ implies
$$b_1 = 0, \ldots, b_n = 0,$$
and every vector $u \in U$ can be written
$$u = b_1 w_1 + \cdots + b_n w_n$$
for some $b_1, \ldots, b_n \in \mathbb{F}$. Therefore, $a_1 u_1 + \cdots + a_m u_m, b_1 w_1 + \cdots + b_n w_n = 0 + 0 = 0$ implies
$$a_1 = 0, \ldots, a_m = 0, b_1 = 0, \ldots, b_n = 0,$$
and so $u_1, \ldots, u_m, w_1, \ldots, w_n$ is linearly independent in V. Also, every vector $v \in V$ can be written
$$\begin{aligned} v &= u + w \\ &= (a_1 u_1 + \cdots + a_m u_m) + (b_1 w_1 + \cdots + b_n w_n) \\ &= a_1 u_1 + \cdots + a_m u_m + b_1 w_1 + \cdots + b_n w_n, \end{aligned}$$
which means $u_1, \ldots, u_m, w_1, \ldots, w_n$ spans V. Therefore, $u_1, \ldots, u_m, w_1, \ldots, w_n$ is a basis of V. □

2.C.1. Suppose V is finite-dimensional and U is a subspace of V such that $\dim U = \dim V$. Prove that we have $U = V$.

Proof. Let $u_1, \ldots, u_n$ be a basis of U, which means we have $n = \dim U$. This means $u_1, \ldots, u_n$ spans U—that is, we have $\operatorname{span}(u_1, \ldots, u_n) = U$—and is linearly independent in U. In fact, since U is a subspace of V, it is also true that $u_1, \ldots, u_n$ is a linearly independent list in V. Since we have $\dim U = \dim V$, it follows that we have $\dim V = n$. So the linearly independent list $u_1, \ldots, u_n$ in V has length n and $\dim V = n$. By 2.39 of Axler, $u_1, \ldots, u_n$ is a basis of V. This means in particular that $u_1, \ldots, u_n$ spans V, which means we have $\operatorname{span}(u_1, \ldots, u_n) = U$. Therefore, we conclude $U = V$, as desired. □

2.C.2. Show that the subspaces of $\mathbb{R}^2$ are precisely $\{0\}$, $\mathbb{R}^2$, and all lines in $\mathbb{R}^2$ through the origin.

Proof. Let U be a subspace of $\mathbb{R}^2$. Since 2.37 of Alxer asserts $\dim \mathbb{R}^2 = 2$, it follows by 2.38 of Axler that $\dim U$ is one of $0, 1, 2$. So we will argue by cases.

- Case 1: Suppose $\dim U = 0$. Notice that the dimension of the trivial set is zero; that is, we have $\dim\{0\} = 0$. Therefore, we have $\dim U = \dim\{0\}$. By Exercise 2.C.1, we conclude $U = \{0\}$.
- Case 2: Suppose $\dim U = 1$. Then the length of the basis of U is 1; in other words, the basis of U contains only one nonzero vector. This means in particular the list contains one nonzero vector that spans all of U; every vector in U is a scalar multiple (linear combination) of the one basis vector. The set of all such vectors in U describes a line in $\mathbb{R}^2$; in other words, U is a line in $\mathbb{R}^2$. Furthermore, as U is a subspace, in particular it contains the additive identity $(0, 0) \in \mathbb{R}^2$. Therefore, U must be a line in $\mathbb{R}^2$ that passes through the origin.
- Case 3: Suppose $\dim U = 2$. Notice that we have $\dim \mathbb{R}^2 = 2$. Therefore, we have $\dim U = \dim \mathbb{R}^2$. By Exercise 2.C.1, we must have $U = \mathbb{R}^2$.

The cases complete our proof. □

2.C.3. Show that the subspaces of $\mathbb{R}^2$ are precisely $\{0\}$, $\mathbb{R}^3$, all lines in $\mathbb{R}^2$ through the origin, and all planes in $\mathbb{R}^3$ through the origin.

Proof. Let U be a subspace of $\mathbb{R}^3$. Since 2.37 of Alxer asserts $\dim \mathbb{R}^2 = 3$, it follows by 2.38 of Axler that $\dim U$ is one of $0, 1, 2, 3$. So we will argue by cases.

- Case 1: Suppose $\dim U = 0$. Notice that the dimension of the trivial set is zero; that is, we have $\dim\{0\} = 0$. Therefore, we have $\dim U = \dim\{0\}$. By Exercise 2.C.1, we conclude $U = \{0\}$.
- Case 2: Suppose $\dim U = 1$. Then the length of the basis of U is 1; in other words, the basis of U contains only one nonzero vector. This means in particular the list contains one nonzero vector that spans all of U; every vector in U is a scalar multiple (linear combination) of the one basis vector. The set of all such vectors in U describes a line in $\mathbb{R}^3$; in other words, U is a line in $\mathbb{R}^3$. Furthermore, as U is a subspace, in particular it contains the additive identity $(0, 0, 0) \in \mathbb{R}^3$. Therefore, U must be a line in $\mathbb{R}^3$ that passes through the origin.
- Case 3: Suppose $\dim U = 2$. Then the length of the basis of U is 2; in other words, the basis of U contains two nonzero vectors. This means in particular the list contains two nonzero vectors that spans all of U; every vector in U is a linear combination of the two basis vectors. The set of all such vectors in U describes a plane in $\mathbb{R}^3$; in other words, U is a plane in $\mathbb{R}^3$. Furthermore, as U is a subspace, in particular it contains the additive identity $(0, 0, 0) \in \mathbb{R}^3$. Therefore, U must be a plane in $\mathbb{R}^3$ that passes through the origin.

- Case 4: Suppose $\dim U = 3$. Notice that we have $\dim \mathbb{R}^3 = 3$. Therefore, we have $\dim U = \dim \mathbb{R}^3$. By Exercise 2.C.1, we must have $U = \mathbb{R}^3$.

The cases complete our proof. □

2.C.11. Suppose that U and W are subspaces of $\mathbb{R}^8$ such that $\dim U = 3$, $\dim W = 5$, and $U + W = \mathbb{R}^8$. Prove that $\mathbb{R}^8 = U \oplus W$.

Proof. We recall from 2.43 of Axler the formula for the dimension of a sum for our subspaces U and W of $\mathbb{R}^8$:

$$\dim(U + W) = \dim U + \dim W - \dim(U \cap W).$$

By 1.39 of Axler, $U + W$ is a subspace of $\mathbb{R}^8$. By 2.38 of Axler, we get $\dim(U + W) \leq 8$. So we have

$$\begin{aligned}\dim(U \cap W) &= \dim U + \dim W - \dim(U + W)\\ &= 3 + 5 - \dim(U + W)\\ &= 8 - \dim(U + W)\\ &\geq 8 - 8\\ &= 0\\ &= \dim\{0\}.\end{aligned}$$

By Exercise 1.C.1, we get $U \cap W = \{0\}$. So we have $U + w = \mathbb{R}^8$ and $U \cap W = \{0\}$, and so by 1.45 of Axler we conclude that $U + W$ is a direct sum, which means we have $\mathbb{R}^8 = U + W = U \oplus W$. □

2.C.12. Suppose U and W are both five-dimensional subspaces of $\mathbb{R}^9$. Prove that we have $U \cap W \neq \{0\}$.

Proof. We recall from 2.43 of Axler the formula for the dimension of a sum for our subspaces U and W of $\mathbb{R}^9$:

$$\dim(U + W) = \dim U + \dim W - \dim(U \cap W).$$

Since U and W are both five-dimensional subspaces of $\mathbb{R}^9$, we have $\dim U = \dim W = 5$. By 1.39 of Axler, $U + W$ is a subspace of $\mathbb{R}^9$. By 2.38 of Axler, we get $\dim(U + W) \leq 9$. So we have

$$\begin{aligned}\dim(U \cap W) &= \dim U + \dim W - \dim(U + W)\\ &= 5 + 5 - \dim(U + W)\\ &= 10 - \dim(U + W)\\ &\geq 10 - 9\\ &= 1.\end{aligned}$$

If we assume by contradiction that we have $U \cap W = \{0\}$, then we would obtain

$$\begin{aligned}1 &= \dim(U \cap W)\\ &= \dim\{0\}\\ &= 0,\end{aligned}$$

which is a contradiction. Therefore, we conclude $U \cap W \neq \{0\}$. □

2.C.13. Suppose U and W are both 4-dimensional subspaces of $\mathbb{C}^6$. Prove that there exist two vectors $U \cap W$ such that neither of these vectors is a scalar multiple of the other.

Proof. We recall from 2.43 of Axler the formula for the dimension of a sum for our subspaces U and W of $\mathbb{R}^9$:

$$\dim(U + W) = \dim U + \dim W - \dim(U \cap W).$$

Since U and W are both four-dimensional subspaces of $\mathbb{C}^6$, we have $\dim U = \dim W = 4$ and, according to Exercise 1.C.10 of Axler, $U \cap W$ is a subspace of $\mathbb{C}^6$. Furthermore, by 2.26 of Axler, $U \cap W$ is finite-dimensional, and so by 2.32 of Axler there exists a basis of $U \cap W$. By 1.39 of Axler, $U + W$ is a subspace of $\mathbb{R}^9$. By 2.38 of Axler, we get $\dim(U + W) \leq 6$. So we have

$$\begin{aligned}\dim(U \cap W) &= \dim U + \dim W - \dim(U + W)\\ &= 4 + 4 - \dim(U + W)\\ &= 8 - \dim(U + W)\\ &\geq 8 - 6\\ &= 2.\end{aligned}$$

Since we establsihed $\dim(U \cap W) \geq 2$, the basis of $U \cap W$ is at least length 2. As the basis of $U \cap W$ is a linearly independent set in $U \cap W$, we can find two of the vectors in the basis, neither of which is a scalar multiple of the other. □

2.C.15. Suppose V is finite-dimensional, with $\dim V = n \geq 1$. Prove that there exist 1-dimensional subspaces $U_1, \ldots, U_n$ of V such that

$$V = U_1 \oplus \cdots \oplus U_n.$$

Proof. Since V is finite-dimensional, by 2.32 of Axler there exists a basis $v_1, \ldots, v_n$ of V. Let $v \in V$ be an arbitrary vector. Then we can write it uniquely in the form

$$v = a_1 v_1 + \cdots + a_n v_n$$

for some $a_1, \ldots, a_n \in \mathbb{F}$. Let $U_i = \operatorname{span}(v_i)$ for each $i = 1, \ldots, n$. Then, since v_i is a list of one vector in V, it follows by 2.7 of Axler that U_i is a subspace of V. By construction, the list v_i spans U_i, which means we can write each vector in U_i of the form $a_i v_i \in U_i$ for some $a_i \in \mathbb{F}$. Furthermore, if $a_i \in \mathbb{F}$ satisfies

$$a_i v_i = 0,$$

then we must have $a_i = 0$ because $v_i \in U_i$ is a nonzero vector, and so the list v_i is linearly independent in U_i. Therefore, v_i is a linearly independent list that spans U_i, which means v_i is a basis of U_i. The length of the basis v_i is 1, so we get $\dim U_i = 1$ for all $i = 1, \ldots, n$. So we conclude that $U_1, \ldots, U_n$ are 1-dimensional subspaces of V. Now, since the sum $U_1 + \cdots + U_n$ consists of all possible sums of elements of $U_1, \ldots, U_n$, we have $v \in U_1 + \cdots + U_n$, and so we obtain the set containment $V \subset U_1 + \cdots + U_n$. However, by 1.39 of Axler, $U_1 + \cdots + U_n$ is a subspace of V. So we have, in fact, the set equality $V = U_1 + \cdots + U_n$. Now, we need to show that the sum $U_1 + \cdots + U_n$ is indeed the direct sum. Consider the vector $v = 0$. Then we have

$$0 = a_1 v_1 + \cdots + a_n v_n$$

for some $a_1, \ldots, a_n \in \mathbb{F}$. Since the list $v_1, \ldots, v_n$ is a basis of V, the criterion for a basis (2.29 of Axler) asserts that the form $a_1 v_1 + \cdots + a_n v_n$ is unique. So the above equation implies that the only way to write the zero vector 0 as a sum of $a_1 v_1 + \cdots + a_n v_n$ is to take each $a_i v_i \in U_i$ to be equal to 0. By 1.44 of Axler, the sum of the subspaces $U_1, \ldots, U_n$ of V is in fact a direct sum; that is, we have $U_1 + \cdots + U_n = U_1 \oplus \cdots \oplus U_n$. Therefore, we conclude $V = U_1 \oplus \cdots \oplus U_n$. □

2.C.16. Suppose $U_1, \ldots, U_m$ are finite-dimensional subspaces of V such that $U_1 + \cdots + U_m$ is a direct sum. Prove that $U_1 \oplus \cdots \oplus U_m$ is finite-dimensional and

$$\dim(U_1 \oplus \cdots \oplus U_m) = \dim U_1 + \cdots + \dim U_m.$$

Proof. Since $U_1 + \cdots + U_m$ is a direct sum, we can write $U_1 + \cdots + U_m = U_1 \oplus \cdots \oplus U_m$. We will use induction to prove the statement

$$\dim(U_1 \oplus \cdots \oplus U_m) = \dim U_1 + \cdots + \dim U_m$$

for all positive integers m.

- Base step: The statement for $m = 1$ is

  $$\dim(U_1 \oplus U_2) = \dim U_1 + \dim U_2,$$

  which we will need to prove. Since $U_1 + U_2$ is a direct sum, by 1.45 of Axler, we get $U_1 \cap U_2 = \{0\}$. Taking dimensions, we get $\dim(U_1 \cap U_2) = \dim\{0\} = 0$. Using the formula for the dimension of a sum (2.43 of Axler), we have

  $$\begin{aligned} \dim(U_1 \oplus U_2) &= \dim(U_1 + U_2) \\ &= \dim U_1 + \dim U_2 - \dim(U_1 \cap U_2) \\ &= \dim U_1 + \dim U_2 - 0 \\ &= \dim U_1 + \dim U_2. \end{aligned}$$

  This proves the statement for $m = 1$.

- Induction step: The statement for $m = k$ is

  $$\dim(U_1 \oplus \cdots \oplus U_k) = \dim U_1 + \cdots + \dim U_k.$$

  We will prove that the statement holds true for $m = k + 1$. Using our result for the base step with two subspaces and our assumption for the induction step, we have

  $$\begin{aligned} \dim(U_1 \oplus \cdots \oplus U_{k+1}) &= \dim((U_1 \oplus \cdots \oplus U_k) \oplus U_{k+1}) \\ &= \dim((U_1 \oplus \cdots \oplus U_k) + U_{k+1}) \\ &= \dim(U_1 \oplus \cdots \oplus U_k) + \dim U_{k+1} \\ &= (\dim U_1 + \cdots + \dim U_k) + \dim U_{k+1} \\ &= \dim U_1 + \cdots + \dim U_{k+1}. \end{aligned}$$

  This proves the statement for $m = k + 1$.

This completes our proof by induction. □

2.C.17. You might guess, by analogy with the formula for the number of elements in the union of three subsets of a finite set, that if U_1, U_2, U_3 are subspaces of a finite-dimensional vector space, then

$$\dim(U_1 + U_2 + U_3) = \dim U_1 + \dim U_2 + \dim U_3 - \dim(U_1 \cap U_2) - \dim(U_1 \cap U_3) - \dim(U_2 \cap U_3) + \dim(U_1 \cap U_2 \cap U_3).$$

Prove this or give a counterexample.

Proof. We will give a counterexample to show that this statement is false. Let $V = \mathbb{R}^2$ be a vector space, and consider its subspaces $U_1 = \{(x_1, 0) \in \mathbb{R}^2 : x_1 \in \mathbb{R}\}$, $U_2 = \{(x_1, x_1) \in \mathbb{R}^2 : x_1 \in \mathbb{R}\}$, and $U_3 = \{(0, x_2) \in \mathbb{R}^2 : x_2 \in \mathbb{R}\}$. Then we have the intersections $U_1 \cap U_2 = U_1 \cap U_3 = U_2 \cap U_3 = U_1 \cap U_2 \cap U_3 = \{(0, 0)\}$ and the sum

$$\begin{aligned} U_1 + U_2 + U_3 &= \{(x_1, 0) + (x_1, x_1) + (0, x_2) \in \mathbb{R}^2 : (x_1, 0) \in U_1, (x_1, x_1) \in U_2, (0, x_2) \in U_3, x_1, x_2 \in \mathbb{R}\} \\ &= \{(2x_1, x_1) + (0, x_2) \in \mathbb{R}^2 : x_1, x_2 \in \mathbb{R}\} \\ &= \{(2x_1, x_1 + x_2) \in \mathbb{R}^2 : x_1, x_2 \in \mathbb{R}\} \\ &= \mathbb{R}^2. \end{aligned}$$

Taking dimensions, we get $\dim(U_1 \cap U_2) = \dim(U_1 \cap U_3) = \dim(U_2 \cap U_3) = \dim(U_1 \cap U_2 \cap U_3) = \dim\{(0, 0)\} = 0$ and $\dim(U_1 + U_2 + U_3) = \dim \mathbb{R}^2 = 2$. If the above "equation" for $\dim(U_1 + U_2 + U_3)$ is true, then we would get

$$\begin{aligned} 2 &= \dim(U_1 + U_2 + U_3) \\ &= \dim U_1 + \dim U_2 + \dim U_3 - \dim(U_1 \cap U_2) - \dim(U_1 \cap U_3) - \dim(U_2 \cap U_3) + \dim(U_1 \cap U_2 \cap U_3) \\ &= 0 + 0 + 0 - 0 - 0 - 0 + 0 \\ &= 0, \end{aligned}$$

which is not a true statement. So the "equation" generally does not hold true. □