**MATH 131: Linear Algebra I**
University of California, Riverside
Homework 3 Solutions
July 15, 2019

Solutions to assigned homework problems from *Linear Algebra Done Right* (third edition) by Sheldon Axler

3.A: 1, 2, 3, 4, 7, 8, 9, 11, 13
3.B: 9, 13, 14, 15, 17, 18, 19, 20, 21, 29

3.A.1. Suppose $b, c \in \mathbb{R}$. Define $T : \mathbb{R}^3 \to \mathbb{R}^2$ by

$$T(x, y, z) = (2x - 4y + 3z + b, 6x + cxyz).$$

Show that $T$ is linear if and only if $b = c = 0$.

*Proof.* Forward direction: If $T$ is linear, then $b = 0$ and $c = 0$. Since $T$ is linear, additivity holds for all $(x, y, z), (\tilde{x}, \tilde{y}, \tilde{z}) \in \mathbb{R}^3$. It would be a good idea for us to choose simple points in $\mathbb{R}^3$ in order to make our computations as simple as possible. If we let $(x, y, z) = (1, 0, 0), (\tilde{x}, \tilde{y}, \tilde{z}) = (0, 1, 1) \in \mathbb{R}^3$, then we have

$$\begin{aligned} T((1, 0, 0) + (0, 1, 1)) &= T(1, 1, 1) \\ &= (2(1) - 4(1) + 3(1) + b, 6(1) + c(1)(1)(1)) \\ &= (1 + b, 6 + c) \end{aligned}$$

and

$$\begin{aligned} T(1, 0, 0) + T(0, 1, 1) &= (2(1) - 4(0) + 3(0) + b, 6(1) + c(1)(0)(0)) + (2(0) - 4(1) + 3(1) + b, 6(0) + c(0)(1)(1)) \\ &= (2 + b, 6) + (-1 + b, 0) \\ &= (1 + 2b, 6). \end{aligned}$$

Since $T$ is linear, additivity of $T$ holds and implies that we have

$$\begin{aligned} (1 + b, 6 + c) &= T((1, 0, 0) + (0, 1, 1)) \\ &= T(1, 0, 0) + T(0, 1, 1) \\ &= (1 + 2b, 6), \end{aligned}$$

from which we can equate the coordinates to obtain the equations $1 + b = 1 + 2b$ and $6 + c = 6$, which imply $b = 0$ and $c = 0$, respectively.
Backward direction: If $b = 0$ and $c = 0$, then $T$ is linear. Suppose $b = 0$ and $c = 0$. Then the map $T : \mathbb{R}^3 \to \mathbb{R}^2$ becomes

$$T(x, y, z) = (2x - 4y + 3z, 6x).$$

We will prove that $T$ is linear.

- Additivity: For all $(x, y, z), (\tilde{x}, \tilde{y}, \tilde{z}) \in \mathbb{R}^3$, we have

$$\begin{aligned} T((x, y, z) + (\tilde{x}, \tilde{y}, \tilde{z})) &= T(x + \tilde{x}, y + \tilde{y}, z + \tilde{z}) \\ &= (2(x + \tilde{x}) - 4(y + \tilde{y}) + 3(z + \tilde{z}), 6(x + \tilde{x})) \\ &= (2x - 4y + 3z + 2\tilde{x} - 4\tilde{y} + 3\tilde{z}, 6x + 6\tilde{x}) \\ &= (2x - 4y + 3z, 6x) + (2\tilde{x} - 4\tilde{y} + 3\tilde{z}, 6\tilde{x}) \\ &= T(x, y, z) + T(\tilde{x}, \tilde{y}, \tilde{z}). \end{aligned}$$

- Homogeneity: For all $\lambda \in \mathbb{F}$ and for all $(x, y, z) \in \mathbb{R}^3$, we have

$$\begin{aligned} T(\lambda(x, y, z)) &= T(\lambda x, \lambda y, \lambda z) \\ &= (2(\lambda x) - 4(\lambda y) + 3(\lambda z), 6(\lambda x)) \\ &= (\lambda(2x - 4y + 3z), \lambda(6x)) \\ &= \lambda(2x - 4y + 3z, 6x) \\ &= \lambda T(x, y, z). \end{aligned}$$

Since additivity and homogeneity of $T$ are satisfied, $T$ is linear. □

3.A.2. Suppose $b, c \in \mathbb{R}$. Define $T : \mathcal{P}(\mathbb{R}) \to \mathbb{R}^2$ by

$$Tp = \left(3p(4) + 5p'(6) + bp(1)p(2), \int_{-1}^{2} x^3 p(x)\, dx + c \sin p(0)\right).$$

Show that $T$ is linear if and only if $b = c = 0$.

*Proof.* Forward direction: If $T$ is linear, then $b = 0$ and $c = 0$. Since $T$ is linear, additivity holds for all $p, q \in \mathcal{P}(\mathbb{R})$. It would be a good idea for us to choose simple polynomials in $\mathcal{P}(\mathbb{R})$ in order to make our computations as simple as possible. Define $p, q \in \mathcal{P}(\mathbb{R})$ by $p(x) = \frac{\pi}{2}$ and $q(x) = \frac{\pi}{2}$ for all $x \in \mathbb{R}$. Then their first-order derivatives are $p'(x) = 0$ and $q'(x) = 0$ for all $x \in \mathbb{R}$, and so we have

$$\begin{aligned}
T(p+q) &= \left(3(p+q)(4) + 5(p+q)'(6) + b(p+q)(1)(p+q)(2), \int_{-1}^{2} x^3 (p+q)(x)\,dx + c\sin((p+q)(0))\right) \\
&= \left(3(p(4)+q(4)) + 5(p'(6)+q'(6)) + b(p(1)+q(1))(p(2)+q(2)), \int_{-1}^{2} x^3 (p(x)+q(x))\,dx + c\sin(p(0)+q(0))\right) \\
&= \left(3\left(\frac{\pi}{2}+\frac{\pi}{2}\right) + 5(0+0) + b\left(\frac{\pi}{2}+\frac{\pi}{2}\right)\left(\frac{\pi}{2}+\frac{\pi}{2}\right), \int_{-1}^{2} x^3 \left(\frac{\pi}{2}+\frac{\pi}{2}\right)dx + c\sin\left(\frac{\pi}{2}+\frac{\pi}{2}\right)\right) \\
&= \left(3\pi + \pi^2 b, \frac{15\pi}{4}\right)
\end{aligned}$$

and

$$\begin{aligned}
Tp + Tq &= \left(3p(4) + 5p'(6) + bp(1)p(2), \int_{-1}^{2} x^3 p(x)\,dx + c\sin p(0)\right) + \left(3q(4) + 5q'(6) + bq(1)q(2), \int_{-1}^{2} x^3 q(x)\,dx + c\sin q(0)\right) \\
&= \left(3\left(\frac{\pi}{2}\right) + 5(0) + b\left(\frac{\pi}{2}\right)\left(\frac{\pi}{2}\right), \int_{-1}^{2} x^3\left(\frac{\pi}{2}\right)dx + c\sin\left(\frac{\pi}{2}\right)\right) + \left(3\left(\frac{\pi}{2}\right) + 5(0) + b\left(\frac{\pi}{2}\right)\left(\frac{\pi}{2}\right), \int_{-1}^{2} x^3\left(\frac{\pi}{2}\right)dx + c\sin\left(\frac{\pi}{2}\right)\right) \\
&= \left(\frac{3\pi}{2} + \frac{\pi b}{4}, \frac{\pi}{2}\int_{-1}^{2} x^3\,dx + c\right) + \left(\frac{3\pi}{2} + \frac{\pi b}{4}, \frac{\pi}{2}\int_{-1}^{2} x^3\,dx + c\right) \\
&= \left(3\pi + \frac{\pi b}{2}, \frac{15\pi}{4} + 2c\right).
\end{aligned}$$

Since $T$ is linear, additivity of $T$ holds and implies that we have

$$\begin{aligned}
\left(3\pi + \pi^2 b, \frac{15\pi}{4}\right) &= T(p+q) \\
&= Tp + Tq \\
&= \left(3\pi + \frac{\pi b}{2}, \frac{15\pi}{4} + 2c\right),
\end{aligned}$$

from which we can equate the coordinates to obtain the equations $3\pi + \pi^2 b = 3\pi + \frac{\pi b}{2}$ and $\frac{15\pi}{4} = \frac{15\pi}{4} + 2c$, which imply $b = 0$ and $c = 0$, respectively.

Backward direction: If $b = 0$ and $c = 0$, then $T$ is linear. Suppose $b = 0$ and $c = 0$. Then the map $T : \mathbb{R}^3 \to \mathbb{R}^2$ becomes

$$Tp = \left(3p(4) + 5p'(6), \int_{-1}^{2} x^3 p(x)\,dx\right).$$

We will prove that $T$ is linear.

- Additivity: For all $p, q \in \mathcal{P}(\mathbb{R})$, we have

$$\begin{aligned}
T(p+q) &= \left(3(p+q)(4) + 5(p+q)'(6), \int_{-1}^{2} x^3 (p+q)(x)\,dx\right) \\
&= \left(3(p(4)+q(4)) + 5(p'(6)+q'(6)), \int_{-1}^{2} x^3 (p(x)+q(x))\,dx\right) \\
&= \left(3p(4) + 5p'(6) + 3q(4) + 3q'(6), \int_{-1}^{2} x^3 p(x)\,dx + \int_{-1}^{2} x^3 q(x)\,dx\right) \\
&= \left(3p(4) + 5p'(6), \int_{-1}^{2} x^3 p(x)\,dx\right) + \left(3q(4) + 3q'(6), \int_{-1}^{2} x^3 q(x)\,dx\right) \\
&= Tp + Tq.
\end{aligned}$$

- Homogeneity: For all $\lambda \in \mathbb{F}$ and for all $(x, y, z) \in \mathbb{R}^3$, we have

$$\begin{aligned}
T(\lambda p) &= \left(3(\lambda p)(4) + 5(\lambda p)'(6), \int_{-1}^{2} x^3 (\lambda p)(x)\, dx\right) \\
&= \left(3\lambda p(4) + 5\lambda p'(6), \int_{-1}^{2} x^3 \lambda p(x)\, dx\right) \\
&= \left(\lambda(3p(4) + 5p'(6)), \lambda \int_{-1}^{2} x^3 p(x)\, dx\right) \\
&= \lambda \left(3p(4) + 5p'(6), \int_{-1}^{2} x^3 p(x)\, dx\right) \\
&= \lambda T p.
\end{aligned}$$

Since additivity and homogeneity of $T$ are satisfied, $T$ is linear. □

3.A.3. Suppose $T \in \mathcal{L}(\mathbb{F}^n, \mathbb{F}^m)$. Show that there exist scalars $A_{j,k} \in \mathbb{F}$ for $j = 1, \ldots, m$ and for $k = 1, \ldots, n$ such that

$$T(x_1, \ldots, x_n) = (A_{1,1}x_1 + \cdots + A_{1,n}x_n, \ldots, A_{m,1}x_1 + \cdots + A_{m,n}x_n)$$

for all $(x_1, \ldots, x_n) \in \mathbb{F}^n$.

*Proof.* Suppose $A_{j,k} \in \mathbb{F}$ for $j = 1, \ldots, m$ and for $k = 1, \ldots, n$ satisfy

$$\begin{aligned}
T(1, 0, \ldots, 0) &= (A_{1,1}, \ldots, A_{m,1}), \\
T(0, 1, 0, \ldots, 0) &= (A_{1,2}, \ldots, A_{m,2}), \\
&\vdots \\
T(0, \ldots, 0, 1) &= (A_{1,n}, \ldots, A_{m,n}).
\end{aligned}$$

Then, for all $(x_1, \ldots, x_n) \in \mathbb{F}^n$, we have

$$\begin{aligned}
T(x_1, \ldots, x_n) &= T((x_1, 0, \ldots, 0) + (0, x_2, 0, \ldots, 0) + \cdots + (0, \ldots, 0, x_n)) \\
&= T(x_1(1, 0, \ldots, 0) + x_2(0, 1, 0, \ldots, 0) + \cdots + x_n(0, \ldots, 0, 1)) \\
&= T(x_1(1, 0, \ldots, 0)) + T(x_2(0, 1, 0, \ldots, 0)) + \cdots + T(x_n(0, \ldots, 0, 1)) \\
&= x_1 T(1, 0, \ldots, 0) + x_2 T(0, 1, 0, \ldots, 0) + \cdots + x_n T(0, \ldots, 0, 1) \\
&= x_1(A_{1,1}, \ldots, A_{m,1}) + x_2(A_{1,2}, \ldots, A_{m,2}) + \cdots + x_n(A_{1,n}, \ldots, A_{m,n}) \\
&= (A_{1,1}x_1, \ldots, A_{m,1}x_1) + (A_{1,2}x_2, \ldots, A_{m,2}x_2) + \cdots + (A_{1,n}x_n, \ldots, A_{m,n}x_n) \\
&= (A_{1,1}x_1 + \cdots + A_{1,n}x_n, \ldots, A_{m,1}x_1 + \cdots + A_{m,n}x_n),
\end{aligned}$$

as desired. □

3.A.4. Suppose $T \in \mathcal{L}(V, W)$ and $v_1, \ldots, v_m$ is a list of vectors in $V$ such that $Tv_1, \ldots, Tv_m$ is a linearly independent list in $W$. Prove that $v_1, \ldots, v_m$ is linearly independent.

*Proof.* Suppose $a_1, \ldots, a_n \in \mathbb{F}$ satisfy

$$a_1 v_1 + \cdots + a_n v_n = 0.$$

Then, since $T$ is linear, we have

$$\begin{aligned}
a_1 T v_1 + \cdots + a_n T v_n &= T(a_1 v_1 + \cdots + a_n v_n) \\
&= T(0) \\
&= 0.
\end{aligned}$$

Since $Tv_1, \ldots, Tv_n$ is linearly independent in $W$, we have

$$a_1 = 0, \ldots, a_n = 0.$$

So $v_1, \ldots, v_n$ is linearly independent in $V$. □

3.A.7. Show that every linear map from a 1-dimensional vector space $V$ to itself is a multiplication by some scalar. More precisely, prove that if $\dim V = 1$ and $T \in \mathcal{L}(V, V)$, then there exists $\lambda \in \mathbb{F}$ such that $Tv = \lambda v$ for all $v \in V$.

*Proof.* Let $u \in V$ be a nonzero vector. Since we assume $\dim V = 1$, it follows that every vector in $V$ is a scalar multiple of you; in other words, for all $v \in V$, there exists $a \in \mathbb{F}$ that satisfies $v = au$. In fact, since we have $Tv \in V$, there exists $\lambda \in \mathbb{F}$ that satisfies $Tv = \lambda u$. As we assume $T \in \mathcal{L}(V, V)$, we can use its homogeneity to obtain

$$\begin{aligned} Tv &= T(au) \\ &= aTu \\ &= a(\lambda u) \\ &= (a\lambda)u \\ &= (\lambda a)u \\ &= \lambda(au) \\ &= \lambda v, \end{aligned}$$

as desired. □

3.A.8. Give an example of a function $\varphi : \mathbb{R}^2 \to \mathbb{R}$ such that

$$\varphi(av) = a\varphi(v)$$

for all $a \in \mathbb{R}$ and for all $v \in \mathbb{R}^2$ but $\varphi$ is not linear.

*Proof.* Since we have $v = \mathbb{R}^2$, it is a list with length 2, and so we can write $v = (x_1, x_2) \in \mathbb{R}^2$. Define, for example, $\varphi : \mathbb{R}^2 \to \mathbb{R}$ by

$$\varphi(x_1, x_2) = (x_1^3 + x_2^3)^{\frac{1}{3}}.$$

Then, for all $a \in \mathbb{R}$ and for all $(x_1, x_2) \in \mathbb{R}^2$, we have

$$\begin{aligned} \varphi(a(x_1, x_2)) &= \varphi(ax_1, ax_2) \\ &= ((ax_1)^3 + (ax_2)^3)^{\frac{1}{3}} \\ &= (a^3x_1^3 + a^3x_2^3)^{\frac{1}{3}} \\ &= (a^3(x_1^3 + x_2^3))^{\frac{1}{3}} \\ &= a(x_1^3 + x_2^3)^{\frac{1}{3}} \\ &= a\varphi(x_1, x_2), \end{aligned}$$

which shows the homogeneity of $\varphi$. However, if we consider the points $(x_1, x_2) = (1, 0), (y_1, y_2) = (0, 1) \in \mathbb{R}^2$, then we have

$$\begin{aligned} \varphi((x_1, x_2) + (y_1, y_2)) &= \varphi((1, 0) + (0, 1)) \\ &= \varphi(1, 1) \\ &= ((1)^3 + (1)^3)^{\frac{1}{3}} \\ &= 2^{\frac{1}{3}} \end{aligned}$$

and

$$\begin{aligned} \varphi(x_1, x_2) + \varphi(y_1, y_2) &= \varphi(1, 0) + \varphi(0, 1) \\ &= ((1)^3 + (0)^3)^{\frac{1}{3}} + ((0)^3 + (1)^3)^{\frac{1}{3}} \\ &= 1 + 1 \\ &= 2. \end{aligned}$$

Since we have $2^{\frac{1}{3}} \neq 2$, we conclude $\varphi((x_1, x_2) + (y_1, y_2)) \neq \varphi(x_1, x_2) + \varphi(y_1, y_2)$, which means that the additivity of $\varphi$ is not satisfied. Therefore, $\varphi$ is not linear. □

3.A.9. Give an example of a function $\varphi : \mathbb{C} \to \mathbb{C}$ such that

$$\varphi(w + z) = \varphi(w) + \varphi(z)$$

for all $w, z \in \mathbb{C}$ but $\varphi$ is not linear.

*Proof.* Define, for example, $\varphi : \mathbb{C} \to \mathbb{C}$ by

$$\varphi(z) = \overline{z},$$

where the bar over $z \in \mathbb{C}$ denotes the complex conjugate of $z$. Since we have $w, z \in \mathbb{C}$, there exist $w_1, w_2, z_1, z_2 \in \mathbb{R}$ that satisfy $w = w_1 + w_2 i$ and $z = z_1 + z_2 i$. So, for all $w, z \in \mathbb{C}$, we have

$$\begin{aligned}
\varphi(w+z) &= \overline{w+z} \\
&= \overline{(w_1 + w_2 i) + (z_1 + z_2 i)} \\
&= \overline{(w_1 + z_1) + (w_2 + z_2) i} \\
&= (w_1 + z_1) - (w_2 + z_2) i \\
&= (w_1 - w_2 i) + (z_1 - z_2 i) \\
&= \overline{w} + \overline{z} \\
&= \varphi(w) + \varphi(z),
\end{aligned}$$

which shows the homogeneity of $\varphi$. However, if we consider $\lambda = i \in \mathbb{F}$ and $z = i \in \mathbb{C}$, then we have

$$\begin{aligned}
\varphi(\lambda z) &= \varphi(ii) \\
&= \varphi(i^2) \\
&= \varphi(-1) \\
&= \overline{-1} \\
&= -1
\end{aligned}$$

and

$$\begin{aligned}
\lambda \varphi(z) &= i \varphi(i) \\
&= i \overline{i} \\
&= i(-i) \\
&= -i^2 \\
&= -(-1) \\
&= 1.
\end{aligned}$$

Since we have $-1 \neq 1$, we conclude $\varphi(\lambda z) \neq \lambda \varphi(z)$, which means that the homogeneity of $\varphi$ is not satisfied. Therefore, $\varphi$ is not linear. □

3.A.11. Suppose $V$ is finite-dimensional. Prove that every linear map on a subspace of $V$ can be extended to a linear map on $V$. In other words, show that if $U$ is a subspace of $V$ and $S \in \mathcal{L}(U, W)$, then there exists $T \in \mathcal{L}(V, W)$ such that $Tu = Su$ for all $u \in U$.

*Proof.* Suppose $U$ is a subspace of $V$, and let $u_1, \ldots, u_m$ be a basis of $U$. Then $u_1, \ldots, u_m$ is linearly independent, and so, by 2.33 of Axler, we can extend this list to a basis $u_1, \ldots, u_m, v_1, \ldots, v_n$ of $V$. This means the list $u_1, \ldots, u_m, v_1, \ldots, v_n$ spans $V$, and so we can write every $v, w \in V$ as unique representations

$$v = a_1 u_1 + \cdots + a_m u_m + b_1 v_1 + \cdots + b_n v_n$$

and

$$w = c_1 u_1 + \cdots + c_m u_m + d_1 v_1 + \cdots + d_n v_n$$

for some $a_1, \ldots, a_m, b_1, \ldots, b_n, c_1, \ldots, c_m, d_1, \ldots, d_n \in \mathbb{F}$. Now, define $T : V \longrightarrow W$ by

$$T(a_1 u_1 + \cdots + a_m u_m + b_1 v_1 + \cdots + b_n v_n) = S(a_1 u_1 + \cdots + a_m u_m).$$

First, given that $S : V \longrightarrow W$ is linear, we will prove that $T : V \longrightarrow W$ is linear.

- Additivity: Let $v, w \in V$ be arbitrary. Since $S$ is linear, we can use its additivity and homogeneity to obtain

$$\begin{aligned}
T(v+w) &= T((a_1 u_1 + \cdots + a_m u_m + b_1 v_1 + \cdots + b_n v_n) + (c_1 u_1 + \cdots + c_m u_m + d_1 v_1 + \cdots + d_n v_n)) \\
&= T((a_1 + c_1) u_1 + \cdots + (a_m + c_m) u_m + (b_1 + d_1) v_1 + \cdots + (b_n + d_n) v_n) \\
&= S((a_1 + c_1) u_1 + \cdots + (a_m + c_m) u_m) \\
&= S((a_1 + c_1) u_1) + \cdots + S((a_m + c_m) u_m) \\
&= (a_1 + c_1) S u_1 + \cdots + (a_m + c_m) S u_m \\
&= (a_1 S u_1 + c_1 S u_1) + \cdots + (a_m S u_m + c_m S u_m) \\
&= (a_1 S u_1 + \cdots + a_m S u_m) + (c_1 S u_1 + \cdots + c_m S u_m) \\
&= (S(a_1 u_1) + \cdots + S(a_m u_m)) + (S(c_1 u_1) + \cdots + S(c_m u_m)) \\
&= S(a_1 u_1 + \cdots + a_m u_m) + S(c_1 u_1 + \cdots + c_m u_m) \\
&= T(a_1 u_1 + \cdots + a_m u_m + b_1 v_1 + \cdots + b_n v_n) + T(c_1 u_1 + \cdots + c_m u_m + d_1 v_1 + \cdots + d_n v_n) \\
&= Tv + Tw.
\end{aligned}$$

- Homogeneity: Let $\lambda \in \mathbb{F}$ and $v \in V$ be arbitrary. Since $S$ is linear, we can use its additivity and homogeneity to obtain

$$\begin{aligned} T(\lambda v) &= T(\lambda(a_1u_1 + \cdots + a_mu_m + b_1v_1 + \cdots + b_nv_n)) \\ &= T((\lambda a_1)u_1 + \cdots + (\lambda a_1)u_m + (\lambda b_1)v_1 + \cdots + (\lambda b_n)v_n) \\ &= S((\lambda a_1)u_1 + \cdots + (\lambda a_m)u_m)) \\ &= S((\lambda a_1)u_1) + \cdots + S((\lambda a_m)u_m)) \\ &= (\lambda a_1)Su_1 + \cdots + (\lambda a_m)Su_m \\ &= \lambda(a_1Su_1 + \cdots + a_mSu_m) \\ &= \lambda(S(a_1u_1) + \cdots + S(a_mu_m)) \\ &= \lambda S(a_1u_1 + \cdots + a_mu_m) \\ &= \lambda T(a_1u_1 + \cdots + a_mu_m + b_1v_1 + \cdots + b_nv_n) \\ &= \lambda Tv. \end{aligned}$$

Since additivity and homogeneity of $T$ are satisfied, $T$ is linear. Next, we need to show that $T$ satisfies $Tu = Su$ for all $u \in U$. Since $u_1, \ldots, u_m$ is a basis of $U$, it spans $U$, and so we can write every $u \in U$ as a unique representation

$$u = a_1u_1 + \cdots + a_mu_m$$

for some $a_1, \ldots, a_m \in \mathbb{F}$. We recall $S \in \mathcal{L}(V, W)$ by assumption, and we established already $T \in \mathcal{L}(V, W)$ and $Tu_i = Su_i$ for each $i = 1, \ldots, m$. So, for all $u \in U$, we have

$$\begin{aligned} Tu &= T(a_1u_1 + \cdots + a_mu_m) \\ &= T(a_1u_1) + \cdots + T(a_mu_m) \\ &= a_1Tu_1 + \cdots + a_mTu_m \\ &= a_1Su_1 + \cdots + a_mSu_m \\ &= S(a_1u_1) + \cdots + S(a_mu_m) \\ &= S(a_1u_1 + \cdots + a_mu_m) \\ &= Su, \end{aligned}$$

as desired. □

3.A.13. Suppose $v_1, \ldots, v_m$ is a linearly dependent list of vectors in $V$. Suppose also that $W \neq \{0\}$. Prove that there exist $w_1, \ldots, w_n \in W$ such that no $T \in \mathcal{L}(V, W)$ satisfies $Tv_k = w_k$ for each $k = 1, \ldots, m$.

*Proof.* Since we assume $W \neq \{0\}$, we can consider a list of nonzero vectors $w_1, \ldots, w_n \in W$, which means we have $w_i \neq 0$ for all $i = 1, \ldots, n$. Suppose by contradiction that $T \in \mathcal{L}(V, W)$ must satisfy $Tv_i = w_i$ for some $i = 1, \ldots, m$. Since $v_1, \ldots, v_m$ is a linearly dependent list of vectors in $V$, there exist $a_1, \ldots, a_m \in \mathbb{F}$, not all zero, that satisfy

$$a_1v_1 + \cdots + a_mv_m = 0.$$

For example, for all $j = 1, \ldots, n$, we can choose some $i \in \{1, \ldots, n\}$ such that $a_i \neq 0$ and $a_j = 0$ if $j \neq i$. The reason for choosing $i$ is because we have $Tv_i = w_i$ from earlier. Also, since $T : V \to W$ is linear, we have $T(0) = 0$, according to 3.11 of Axler. Therefore, we have

$$\begin{aligned} w_i &= T(v_i) \\ &= T(0v_1 + \cdots + 0v_{i-1} + 1v_i + 0v_{i+1} + \cdots + 0v_m) \\ &= T(a_1v_1 + \cdots + a_{i-1}v_{i-1} + a_iv_i + a_{i+1}v_{i+1} + \cdots + a_mv_m) \\ &= T(a_1v_1 + \cdots + a_mv_m) \\ &= T(0) \\ &= 0, \end{aligned}$$

which contradicts our assumption $w_i \neq 0$ for all $i = 1, \ldots, n$ at the beginning of this proof. Therefore, if $w_1, \ldots, w_n \in W$ is a list of nonzero vectors, then no $T \in \mathcal{L}(V, W)$ satisfies $Tv_k = w_k$ for each $k = 1, \ldots, m$. □

3.B.9. Suppose $T \in \mathcal{L}(V, W)$ is injective and $v_1, \ldots, v_n$ is linearly independent in $V$. Prove that $Tv_1, \ldots, Tv_n$ is linearly independent in $W$.

*Proof.* Suppose $a_1, \ldots, a_n \in \mathbb{F}$ satisfy $a_1Tv_1 + \cdots + a_nTv_n = 0$. Then, since $T$ is linear, we have

$$\begin{aligned} T(a_1v_1 + \cdots + a_nv_n) &= a_1Tv_1 + \cdots + a_nTv_n \\ &= 0. \end{aligned}$$

Since $T$ is injective, by 3.16 of Axler we have $\operatorname{null} T = \{0\}$, and so we get

$$a_1 v_1 + \cdots + a_n v_n = 0.$$

Finally, since $v_1, \ldots, v_n$ is linearly independent in $V$ by assumption, all the scalars are zero; that is, we have

$$a_1 = 0, \ldots, a_m = 0.$$

Therefore, $Tv_1, \ldots, Tv_n$ is linearly independent in $W$. □

3.B.13. Suppose $T : \mathbb{F}^4 \to \mathbb{F}^2$ is a linear map such that

$$\operatorname{null} T = \{(x_1, x_2, x_3, x_4) \in \mathbb{F}^4 : x_1 = 5x_2 \text{ and } x_3 = 7x_4\}.$$

Prove that $T$ is surjective.

*Proof.* First, we need to find a basis of $\operatorname{null} T = \{(x_1, x_2, x_3, x_4) \in \mathbb{F}^4 : x_1 = 5x_2 \text{ and } x_3 = 7x_4\}$. Let $(x_1, x_2, x_3, x_4) \in \operatorname{null} T$ be arbitrary. Then we have $x_1 = 5x_2$ and $x_3 = 7x_4$, and so we can write

$$\begin{aligned}(x_1, x_2, x_3, x_4) &= (5x_2, x_2, 7x_4, x_4) \\ &= (5x_2, x_2, 0, 0) + (0, 0, 7x_4, x_4) \\ &= x_2(5, 1, 0, 0) + x_4(0, 0, 7, 1).\end{aligned}$$

Since we have $x_2, x_4 \in \mathbb{F}$, we have established that the list $(5, 1, 0, 0), (0, 0, 7, 1)$ spans $\operatorname{null} T$. If we can also show that the list is also linearly independent in $\operatorname{null} T$, then it would in fact be a basis of $\operatorname{null} T$. Suppose $a_1, a_2, a_3 \in \mathbb{F}$ satisfy

$$a_1(5, 1, 0, 0) + a_2(0, 0, 7, 1) = (0, 0, 0, 0).$$

Applying addition and scalar multiplication in $\mathbb{F}^4$ to the left-hand side of the above equation, we get

$$(5a_1, a_1, 7a_2, a_2) = (0, 0, 0, 0),$$

from which we can equate the second and fourth coordinates of both sides to obtain

$$a_1 = 0, a_2 = 0$$

and so the list $(5, 1, 0, 0), (0, 0, 7, 1)$ is linearly independent in $\operatorname{null} T$. So this list is a basis of $\operatorname{null} T$, which means we have $\dim \operatorname{null} T = 2$. By the Fundamental Theorem of Linear Maps (3.22 of Axler), we have

$$\begin{aligned}\dim \operatorname{range} T &= \dim \mathbb{F}^4 - \dim \operatorname{null} T \\ &= 4 - 2 \\ &= 2 \\ &= \dim \mathbb{F}^2.\end{aligned}$$

By Exercise 2.C.1 of Axler, we conclude $\operatorname{range} T = \mathbb{F}^2$, which means $T : \mathbb{F}^4 \to \mathbb{F}^2$ is surjective. □

3.B.14. Suppose $U$ is a 3-dimensional subspace of $\mathbb{R}^8$ and that $T : \mathbb{R}^8 \to \mathbb{R}^5$ is a liner map such that $\operatorname{null} T = U$. Prove that $T$ is surjective.

*Proof.* Since $U$ is a 3-dimensional subspace of $\mathbb{R}^8$, we have $\dim U = 3$. Furthermore, since we assumed $\operatorname{null} T = U$, we have in fact $\dim \operatorname{null} T = \dim U = 3$. By the Fundamental Theorem of Linear Maps (3.22 of Axler), we have

$$\begin{aligned}\dim \operatorname{range} T &= \dim \mathbb{R}^8 - \dim \operatorname{null} T \\ &= 8 - 3 \\ &= 5 \\ &= \dim \mathbb{R}^5.\end{aligned}$$

By Exercise 2.C.1 of Axler, we conclude $\operatorname{range} T = \mathbb{R}^5$, which means $T : \mathbb{R}^8 \to \mathbb{R}^5$ is surjective. □

3.B.15. Prove that there does not exist a linear map from $\mathbb{F}^5$ to $\mathbb{F}^2$ whose null space equals

$$\{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{F}^5 : x_1 = 3x_2 \text{ and } x_3 = x_4 = x_5\}.$$

*Proof.* Suppose by contradiction there exists $T \in \mathcal{L}(\mathbb{F}^5, \mathbb{F}^2)$ that satisfies

$$\operatorname{null} T = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{F}^5 : x_1 = 3x_2 \text{ and } x_3 = x_4 = x_5\}.$$

First, we need to find a basis of $\operatorname{null} T = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{F}^5 : x_1 = 3x_2 \text{ and } x_3 = x_4 = x_5\}$. Let $(x_1, x_2, x_3, x_4, x_5) \in \operatorname{null} T$ be arbitrary. Then we have $x_1 = 3x_2$ and $x_3 = x_4 = x_5$, and so we can write

$$\begin{aligned}(x_1, x_2, x_3, x_4, x_5) &= (3x_2, x_2, x_3, x_3, x_3)\\ &= (3x_2, x_2, 0, 0, 0) + (0, 0, x_3, x_3, x_3)\\ &= x_2(3, 1, 0, 0, 0) + x_3(0, 0, 1, 1, 1).\end{aligned}$$

Since we have $x_2, x_3 \in \mathbb{F}$, we have established that the list $(3, 1, 0, 0, 0), (0, 0, 1, 1, 1)$ spans $\operatorname{null} T$. If we can also show that the list is also linearly independent in $\operatorname{null} T$, then it would in fact be a basis of $\operatorname{null} T$. Suppose $a_1, a_2, a_3 \in \mathbb{F}$ satisfy

$$a_1(3, 1, 0, 0, 0) + a_2(0, 0, 1, 1, 1) = (0, 0, 0, 0, 0).$$

Applying addition and scalar multiplication in $\mathbb{F}^5$ to the left-hand side of the above equation, we get

$$(3a_1, a_1, a_2, a_2, a_2) = (0, 0, 0, 0, 0),$$

from which we can equate the second and third coordinates of both sides to obtain

$$a_1 = 0, a_2 = 0$$

and so the list $(3, 1, 0, 0, 0), (0, 0, 1, 1, 1)$ is linearly independent in $\operatorname{null} T$. So this list is a basis of $\operatorname{null} T$, which means we have $\dim \operatorname{null} T = 2$. Since we have $T \in \mathcal{L}(\mathbb{F}^5, \mathbb{F}^2)$, it follows, by 3.14 of Axler, that $\operatorname{range} T$ is a subspace of $\mathbb{F}^2$, and so, by 2.38 of Axler, we must have $\dim \operatorname{range} T \leq 2$. By the Fundamental Theorem of Linear Maps (3.22 of Axler), we have

$$\begin{aligned}\dim \operatorname{null} T &= \dim \mathbb{F}^5 - \dim \operatorname{range} T\\ &\geq 5 - 2\\ &= 3\\ &> 2\\ &= \dim \operatorname{null} T,\end{aligned}$$

which is a contradiction. So we conclude that there does not exist a linear map $T \in \mathcal{L}(\mathbb{F}^5, \mathbb{F}^2)$ that satisfies our claim at the very beginning of this proof. □

3.B.17. Suppose $V$ and $W$ are both finite-dimensional. Prove that there exists an injective linear map from $V$ to $W$ if and only if $\dim V \leq \dim W$.

*Proof.* Forward direction: If there exists a injective linear map $T \in \mathcal{L}(V, W)$, then $\dim V \leq \dim W$. Suppose there exists a injective linear map $T \in \mathcal{L}(V, W)$, which means by 3.16 of Axler we have $\operatorname{null} T = \{0\}$. Since 3.19 of Axler says that $\operatorname{range} T$ is a subspace of $W$, by 2.38 of Axler, we have $\dim \operatorname{range} T \leq \dim W$. By the Fundamental Theorem of Linear Maps (3.22 of Axler), we have

$$\begin{aligned}\dim V &= \dim \operatorname{null} T + \dim \operatorname{range} T\\ &= \dim\{0\} + \dim \operatorname{range} T\\ &= 0 + \dim \operatorname{range} T\\ &= \dim \operatorname{range} T\\ &\leq \dim W,\end{aligned}$$

or $\dim W \leq \dim V$, as desired.

Backward direction: If $\dim V \leq \dim W$, then there exists an injective linear map $T \in \mathcal{L}(V, W)$. Suppose we have $\dim V \leq \dim W$. Since $V$ and $W$ are finite-dimensional, according to 2.32 of Axler, there exist a basis of $V$ and a basis of $W$. For brevity in notation, let $m = \dim W$ and $n = \dim V$, which means $n \leq m$. Define $T : V \to W$ by

$$T(a_1 v_1 + \cdots + a_n v_n) = a_1 w_1 + \cdots + a_m w_m$$

for some $a_1, \ldots, a_n, \ldots, a_m \in \mathbb{F}$. Then $T$ is linear and indeed defines a function, according to the proof for 3.5 in Axler. Now suppose we have $a_1 v_1 + \cdots + a_n v_n \in \operatorname{null} T$. Then we have $T(a_1 v_1 + \cdots + a_n v_n) = 0$, or

$$a_1 w_1 + \cdots + a_m w_m = 0.$$

Since $w_1, \ldots, w_m$ is a basis of $W$, it is linearly independent, which means all the scalars are zero; that is, we have $a_1 = 0, \ldots, a_m = 0$. Since $n \leq m$, we have in particular the first $n$ of the $m$ scalars are zero; that is, we have $a_1 = 0, \ldots, a_m = 0$. So we have

$$a_1 v_1 + \cdots + a_n v_n = 0,$$

which means we have $\operatorname{null} T \subset \{0\}$. But 3.14 of Axler says that $\operatorname{null} T$ is a subspace in $V$, which means in particular that $\operatorname{null} T$ contains the additive identity, or $\{0\} \subset \operatorname{null} T$. Therefore, we have the set equality $\operatorname{null} T = \{0\}$. Finally, by 3.16 of Axler, $T$ is injective. □

3.B.18. Suppose $V$ and $W$ are both finite-dimensional. Prove that there exists a surjective linear map from $V$ to $W$ if and only if $\dim V \geq \dim W$.

*Proof.* Forward direction: If there exists a surjective linear map $T \in \mathcal{L}(V, W)$, then $\dim V \geq \dim W$. Suppose there exists a surjective map $T \in \mathcal{L}(V, W)$, which means we have $\operatorname{range} T = W$, and so $\dim \operatorname{range} T = \dim W$. Since $T$ is a linear map, by 3.11 of Axler we have $T(0) = 0$. So we have $\{0\} \subset \operatorname{null} T$, and so, by 2.38 of Axler, we have $0 = \dim\{0\} \leq \dim \operatorname{null} T$. By the Fundamental Theorem of Linear Maps (3.22 of Axler), we have

$$\begin{aligned} \dim V &= \dim \operatorname{null} T + \dim \operatorname{range} T \\ &= \dim \operatorname{null} T + \dim W \\ &\geq \dim\{0\} + \dim W \\ &= 0 + \dim W \\ &= \dim W, \end{aligned}$$

as desired.

Backward direction: If $\dim V \geq \dim W$, then there exists a surjective map $T \in \mathcal{L}(V, W)$. Suppose we have $\dim V \leq \dim W$. Since $V$ and $W$ are finite-dimensional, according to 2.32 of Axler, there exist a basis of $V$ and a basis of $W$. For brevity in notation, let $m = \dim W$ and $n = \dim V$, which means $n \geq m$. Define $T : V \to W$ by

$$T(a_1 v_1 + \cdots + a_n v_n) = a_1 w_1 + \cdots + a_m w_m$$

for some $a_1, \ldots, a_n, \ldots, a_m \in \mathbb{F}$. Then $T$ is linear and indeed defines a function, according to the proof for 3.5 in Axler. Since $w_1, \ldots, w_n$ is a basis of $W$, every vector in $W$ is a linear combination of $w_1, \ldots, w_n$ and can therefore be written $a_1 w_1 + \cdots + a_m w_m$. This implies that we have $\operatorname{range} T = W$, and so $T$ is surjective. □

3.B.19. Suppose $V$ and $W$ are finite-dimensional and that $U$ is a subspace of $V$. Prove that there exists $T \in \mathcal{L}(V, W)$ such that $\operatorname{null} T = U$ if and only if $\dim U \geq \dim V - \dim W$.

*Proof.* Forward direction: If there exists $T \in \mathcal{L}(V, W)$ such that $\operatorname{null} T = U$, then $\dim U \geq \dim V - \dim W$. Since 3.19 of Axler says that $\operatorname{range} T$ is a subspace of $W$, by 2.38 of Axler, we have $\dim \operatorname{range} T \leq \dim W$. By the Fundamental Theorem of Linear Maps (3.22 of Axler), we have

$$\begin{aligned} \dim V &= \dim \operatorname{null} T + \dim \operatorname{range} T \\ &= \dim U + \dim \operatorname{range} T \\ &\leq \dim U + \dim W, \end{aligned}$$

from which we get $\dim U \geq \dim V - \dim W$.

Backward direction: If $\dim U \geq \dim V - \dim W$, then there exists $T \in \mathcal{L}(V, W)$ such that $\operatorname{null} T = U$. Let $u_1, \ldots, u_m$ be a basis of $U$. Then it is a linearly independent list in $U$, and so, by 2.33 of Axler, we can extend the list to a basis $u_1, \ldots, u_m, v_1, \ldots, v_n$ of $V$. This means we have $\dim U = m$ and $\dim V = m + n$. So every vector $v, \tilde{v} \in V$ can be written as unique representations

$$v = a_1 u_1 + \cdots + a_m u_m + b_1 v_1 + \cdots + b_n v_n$$

and

$$\tilde{v} = c_1 u_1 + \cdots + c_m u_m + d_1 v_1 + \cdots + d_n v_n$$

for some $a_1, \ldots, a_m, b_1, \ldots, b_n, c_1, \ldots, c_m, d_1, \ldots, d_n \in \mathbb{F}$. (The only purpose of introducing $\tilde{v}$ here is to help show that the map $T$ satisfies the additivity property of a linear map.) Let $w_1, \ldots, w_p$ be a basis of $W$; this means we have $\dim W = p$. Define $T : V \to W$ by

$$T(a_1 u_1 + \cdots + a_m u_m + b_1 v_1 + \cdots + b_n v_n) = b_1 w_1 + \cdots + b_n w_n.$$

Since we assumed $\dim U \geq \dim V - \dim W$, we have

$$\begin{aligned} n &= (m + n) - m \\ &= \dim V - \dim U \\ &= (\dim V - \dim W) + \dim W - \dim U \\ &\leq \dim U + \dim W - \dim U \\ &= \dim W \\ &= p, \end{aligned}$$

which means that we have the scalars $w_1, \ldots, w_n, \ldots, w_p$ for all integers $n = 1, \ldots, p$, and so the map $T$ that we just defined above makes sense. Now, we will prove that $T$ is linear.

- Additivity: For all $v, \tilde{v} \in V$, we have

$$\begin{aligned} T(v+\tilde{v}) &= T((a_1u_1+\cdots+a_mu_m+b_1v_1+\cdots+b_nv_n)+(c_1u_1+\cdots+c_mu_m+d_1v_1+\cdots+d_nv_n)) \\ &= T((a_1+c_1)u_1+\cdots+(a_m+c_m)u_m+(b_1+d_1)v_1+\cdots+(b_n+d_n)v_n) \\ &= (b_1+d_1)w_1+\cdots+(b_n+d_n)w_n \\ &= (b_1w_1+d_1w_1)+\cdots+(b_nw_n+d_nw_n) \\ &= (b_1w_1+\cdots+b_nw_n)+(d_1w_1+\cdots+d_nw_n) \\ &= Tv+T\tilde{v}. \end{aligned}$$

- Homogeneity: For all $\lambda \in \mathbb{F}$ and for all $v \in V$, we have

$$\begin{aligned} T(\lambda v) &= T(\lambda(a_1u_1+\cdots+a_mu_m+b_1v_1+\cdots+b_nv_n)) \\ &= T((\lambda a_1)u_1+\cdots+(\lambda a_m)u_m+(\lambda b_1)v_1+\cdots+(\lambda b_n)v_n) \\ &= (\lambda b_1)w_1+\cdots+(\lambda b_n)w_n \\ &= \lambda(b_1w_1)+\cdots+\lambda(b_nw_n) \\ &= \lambda(b_1w_1+\cdots+b_nw_n) \\ &= \lambda T(a_1u_1+\cdots+a_mu_m+b_1v_1+\cdots+b_nv_n) \\ &= \lambda Tv. \end{aligned}$$

Since additivity and homogeneity of $T$ are satisfied, $T$ is linear. Next, we need to prove $\operatorname{null} T = U$. Let $u \in \operatorname{null} T$. Since $u_1, \ldots, u_m$ is a basis of $U$ and $u_1, \ldots, u_m, v_1, \ldots, v_n$ is a basis of $V$, we can write $u$ as its unique representation

$$u = a_1u_1+\cdots+a_mu_m+b_1v_1+\cdots+b_nv_n$$

for some $a_1, \ldots, a_m, b_1, \ldots, b_n \in \mathbb{F}$. Suppose we have $v \in \operatorname{null} T$. Then $Tv = 0$, and so we have

$$\begin{aligned} 0 &= Tu \\ &= T(a_1u_1+\cdots+a_mu_m+b_1v_1+\cdots+b_nv_n) \\ &= b_1w_1+\cdots+b_nw_n. \end{aligned}$$

Since $w_1, \ldots, w_p$ is a basis of $W$, we can write the zero vector $0 \in W$ in the unique representation

$$\begin{aligned} 0 &= b_1w_1+\cdots+b_pw_p \\ &= b_1w_1+\cdots+b_nw_n+b_{n+1}w_{n+1}+\cdots+b_pw_p. \end{aligned}$$

According to the criterion for basis (2.29 of Axler), the representation of the zero vector is unique. So all the scalars must be zero; that is, we have

$$b_1 = 0, \ldots, b_p = 0.$$

In particular, the first $n$ of the $p$ scalars is zero:

$$b_1 = 0, \ldots, b_n = 0.$$

Therefore, we have

$$\begin{aligned} u &= a_1u_1+\cdots+a_mu_m+b_1v_1+\cdots+b_nv_n \\ &= a_1u_1+\cdots+a_mu_m+0v_1+\cdots+0v_n \\ &= a_1u_1+\cdots+a_mu_m. \end{aligned}$$

Finally, since again $u_1, \ldots, u_m$ is a basis of $U$, it spans $U$. So we have $\operatorname{span}(u_1, \ldots, u_m) = U$; that is, $U$ consists of all linear combinations of the list $u_1, \ldots, u_m$. We just wrote $u$ as one such linear combination of $u_1, \ldots, u_m$. So we must have $u \in U$, and so we conclude $\operatorname{null} T \subset U$. Now, we will prove the other set containment. Conversely, suppose we have $u \in U$. Since $u_1, \ldots, u_m$ is a basis of $U$ and $u_1, \ldots, u_m, v_1, \ldots, v_n$ is a basis of $V$, we can write $u$ as its unique representation

$$\begin{aligned} u &= a_1u_1+\cdots+a_mu_m \\ &= a_1u_1+\cdots+a_mu_m+0v_1+\cdots+0v_n \end{aligned}$$

for some $a_1, \ldots, a_m \in \mathbb{F}$. Exactly like in the proof of the other set containment, we can also write the zero vector $0 \in V$ in its unique representation

$$0 = 0u_1+\cdots+0u_m+0v_1+\cdots+0v_n.$$

So we have

$$\begin{aligned} Tu &= T(u+0) \\ &= T((a_1u_1+\cdots+a_mu_m+0v_1+\cdots+0v_n)+(0u_1+\cdots+0u_m+0v_1+\cdots+0v_n)) \\ &= T((a_1+0)u_1+\cdots+(a_m+0)u_m+(0+0)v_1+\cdots+(0+0)v_n) \\ &= T(a_1u_1+\cdots+a_mu_m+0v_1+\cdots+0v_n) \\ &= 0w_1+\cdots+0w_n \\ &= 0. \end{aligned}$$

Therefore, we have $u \in \operatorname{null} T$, and so we obtain $U \subset \operatorname{null} T$. Therefore, we obtain the set equality $\operatorname{null} T = U$. □

3.B.20. Suppose $W$ is finite-dimensional and $T \in \mathcal{L}(V, W)$. Prove that $T$ is injective if and only if there exists $S \in \mathcal{L}(W, V)$ such that $ST$ is the identity map on $V$.

*Proof.* Forward direction: If $T$ is injective, then there exists $S \in \mathcal{L}(W, V)$ such that $ST$ is the identity map on $V$. Define $\tilde{S} : \operatorname{range} T \to V$ by

$$\tilde{S}(Tv) = v.$$

Since $T$ is injective, we have the implication: if $u, v \in V$ satisfy $Tu = Tv$, then we have $u = v$. In other words, two distinct representatives of the same input element in range $T$ implies two distinct representatives of the same output element in $V$, and so we conclude that $T$ is well-defined on range $T$. This also implies that $\tilde{S} : \operatorname{range} T \to V$ is indeed a map. Now, assuming in the premises that $T : V \to W$ is linear, we will show that $\tilde{S}$ is linear on range $T$.

- Additivity: For all $v, \tilde{v} \in V$, the additivity of $T$ implies that we have

$$\begin{aligned} \tilde{S}(Tu + Tv) &= \tilde{S}(T(u + v)) \\ &= u + v \\ &= \tilde{S}(Tu) + \tilde{S}(Tv). \end{aligned}$$

- Homogeneity: For all $\lambda \in \mathbb{F}$ and for all $v \in V$, the homogeneity of $T$ implies that we have

$$\begin{aligned} \tilde{S}(\lambda(Tv)) &= \tilde{S}(T(\lambda v)) \\ &= \lambda v \\ &= \lambda \tilde{S}(Tv). \end{aligned}$$

Since additivity and homogeneity of $\tilde{S}$ are satisfied, $\tilde{S}$ is linear on range $T$. Now, we also recall from 3.19 of Axler that range $T$ is a subspace of $W$. So now we can invoke Exercise 3.A.11 of Axler to extend the linear map $\tilde{S}$ on range $T$ to a linear map $S$ on $W$ such that $Sv = \tilde{S}v$ for all $v \in \operatorname{range} T$. Finally, for all $v \in V$, we have

$$\begin{aligned} (ST)v &= S(Tv) \\ &= \tilde{S}(Tv) \\ &= v, \end{aligned}$$

which means we conclude $ST = I_V$, where $I_V$ is an identity map on $V$, as desired.

Backward direction: If there exists $S \in \mathcal{L}(W, V)$ such that $ST$ is the identity map on $V$, then $T$ is injective. Suppose there exists $S \in \mathcal{L}(W, V)$ such that $ST$ is the identity map on $V$. Then we have $(ST) = I_V$, where $I_V$ is an identity map on $V$. In other words, we have $(ST)v = v$ for all $v \in V$. Now, suppose $u, v \in V$ satisfy $Tu = Tv$. Then we have

$$\begin{aligned} u &= I_V u \\ &= (ST)u \\ &= S(Tu) \\ &= S(Tv) \\ &= (ST)v \\ &= I_V v \\ &= v. \end{aligned}$$

Therefore, $T$ is injective. □

3.B.21. Suppose $V$ is finite-dimensional and $T \in \mathcal{L}(V, W)$. Prove that $T$ is surjective if and only if there exists $S \in \mathcal{L}(W, V)$ such that $TS$ is the identity map on $W$.

*Proof.* Forward direction: If $T$ is surjective, then there exists $S \in \mathcal{L}(W, V)$ such that $TS$ is the identity map on $V$. Since $T : V \to W$ is surjective, we have range $T = W$. Furthermore, by the Fundamental Theorem of Linear Maps (3.22 of Axler), we have that range $T$ is finite-dimensional, or equivalently $W$ is finite-dimensional. So we have that both $V$ and $W$ are finite-dimensional. By 2.32 of Axler, there exists a basis $w_1, \ldots, w_m$ of $W$. So every vector $w \in W$ can be written as its unique representation

$$w = a_1 w_1 + \cdots + a_m w_m$$

for some $a_1, \ldots, a_m \in \mathbb{F}$. Since $T : V \to W$ is surjective, given $w_i$ for each $i = 1, \ldots, m$, there exists $v_i \in V$ that satisfies $Tv_i = w_i$. Now define the map $S : W \to V$ by

$$S(a_1 w_1 + \cdots + a_m w_m) = a_1 v_1 + \cdots + a_m v_m.$$

Then we have

$$\begin{aligned}(TS)(a_1w_1+\cdots+a_mw_m) &= T(S(a_1w_1+\cdots+a_mw_m))\\ &= T(a_1v_1+\cdots+a_mv_m)\\ &= a_1w_1+\cdots+a_mw_m,\end{aligned}$$

and so we have $TS = I_W$, where $I$ is the identity map on $W$.

Backward direction: If there exists $S \in \mathcal{L}(W,V)$ such that $TS$ is the identity map on $V$, then $T$ is surjective. Suppose that there exists $S \in \mathcal{L}(W,V)$ such that $TS$ is the identity map on $V$. Then, for all $w \in W$, we have

$$\begin{aligned}w &= I_W w\\ &= (TS)w\\ &= T(Sw).\end{aligned}$$

Now, $S \in \mathcal{L}(W,V)$ implies that we have $Sw \in V$. So the above equation gives us $w \in \operatorname{range} T$, and so we get $W \subset \operatorname{range} T$. At the same time, 3.19 of Axler tells us that $\operatorname{range} T$ is a subspace of $W$. Therefore, we obtain the set equality $\operatorname{range} T = W$. This means $T$ is surjective. □

3.B.29. Suppose $\varphi \in \mathcal{L}(V,\mathbb{F})$. Suppose $u \in V$ is not in $\operatorname{null}\varphi$. Prove that

$$V = \operatorname{null}\varphi \oplus \{au : a \in \mathbb{F}\}.$$

*Proof.* First, we need to prove

$$V = \operatorname{null}\varphi + \{au : a \in \mathbb{F}\}.$$

Now, the map $\varphi \in \mathcal{L}(V,\mathbb{F})$ implies that every output is a scalar; in particular, we have $\varphi(u), \varphi(v) \in \mathbb{F}$. Furthermore, division of two elements in $\mathbb{F}$ is again an element in $\mathbb{F}$; in this case, $\varphi(u), \varphi(v) \in \mathbb{F}$ imply $\frac{\varphi(v)}{\varphi(u)} \in \mathbb{F}$, from which we can conclude $\frac{\varphi(v)}{\varphi(u)}u \in \{au : a \in \mathbb{F}\}$. Also, observe that we can write every $v \in V$ as

$$v = \left(v - \frac{\varphi(v)}{\varphi(u)}u\right) + \frac{\varphi(v)}{\varphi(u)}u.$$

Since we assumed $\varphi \in \mathcal{L}(V,\mathbb{F})$, we can use its additivity and homogeneity to obtain

$$\begin{aligned}\varphi\left(v - \frac{\varphi(v)}{\varphi(u)}u\right) &= \varphi(v) + \varphi\left(-\frac{\varphi(v)}{\varphi(u)}u\right)\\ &= \varphi(v) - \frac{\varphi(v)}{\varphi(u)}\varphi u\\ &= \varphi(v) - \varphi(v)\\ &= 0,\end{aligned}$$

from which we conclude $v - \frac{\varphi v}{\varphi u}u \in \operatorname{null}\varphi$. Therefore, for all $v \in V$, we conclude

$$\begin{aligned}v &= \left(v - \frac{\varphi(v)}{\varphi(u)}u\right) + \frac{\varphi(v)}{\varphi(u)}u\\ &\in \operatorname{null}\varphi + \{au : a \in \mathbb{F}\},\end{aligned}$$

and so we can write $V = \operatorname{null}\varphi + \{au : a \in \mathbb{F}\}$, as we initially claimed. Next, we need to prove

$$\operatorname{null}\varphi \cap \{au : a \in \mathbb{F}\} = \{0\}.$$

Suppose we have $v \in \operatorname{null}\varphi \cap \{au : a \in \mathbb{F}\}$. Then we have $v \in \operatorname{null}\varphi$ and $v \in \{au : a \in \mathbb{F}\}$. In other words, we have $\varphi(v) = 0$ and $v = au$ for some $a \in \mathbb{F}$. Therefore, we have

$$\begin{aligned}0 &= \varphi(v)\\ &= \varphi(au)\\ &= a\varphi(u).\end{aligned}$$

Since we assumed $u \notin \operatorname{null}\varphi$, it follows that we have $\varphi(u) \neq 0$. So the above equation implies that we must have $a = 0$. In turn, we get

$$\begin{aligned}v &= au\\ &= 0 \cdot u\\ &= 0,\end{aligned}$$

and so we have $v \in \{0\}$. So we conclude $\operatorname{null}\varphi \cap \{au : a \in \mathbb{F}\} \subset \{0\}$. At the same time, 3.14 of Axler states that $\operatorname{null}\varphi$ is a subspace of $V$, which implies in particular that we have $0 \in \operatorname{null}\varphi$. And, of course, $0 \in \mathbb{F}$ satisfies $0 = 0u$, which means we have $0 \in \{au : a \in \mathbb{F}\}$. So we have $0 \in \operatorname{null}\varphi \cap \{au : a \in \mathbb{F}\}$, and so we get $\{0\} \subset \operatorname{null}\varphi \cap \{au : a \in \mathbb{F}\}$. Therefore, we conclude the set equality $\{au : a \in \mathbb{F}\} = \{0\}$, as we initially claimed. Finally, by 1.45 of Axler, we conclude $V = \operatorname{null}\varphi \oplus \{au : a \in \mathbb{F}\}$, as desired. □