MATH 131: Linear Algebra I
University of California, Riverside
Homework 4 Solutions
July 22, 2019

Solutions to assigned homework problems from *Linear Algebra Done Right* (third edition) by Sheldon Axler

3.C: 10, 11, 13, 14, 15
3.D: 1, 9, 10, 11, 13, 14, 16, 20

3.C.10. Suppose A is an $m \times n$ matrix and C is an $n \times p$ matrix. Prove that

$$(AC)_{j,\cdot} = A_{j,\cdot}C$$

for all $j = 1, \ldots, m$. In other words, show that row j of AC equals (row j of A) times C.

Proof. Let m, n, p be positive integers, and suppose we have $A_{i,j}, C_{j,k} \in \mathbb{F}$ for any $i = 1, \ldots, m$, for any $j = 1, \ldots, n$, and for any $k = 1, \ldots, p$. Since A is an $m \times n$ matrix and C is an $n \times p$ matrix, we can write

$$A = \begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & \ddots & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{pmatrix}$$

and

$$C = \begin{pmatrix} C_{1,1} & \cdots & C_{1,p} \\ \vdots & \ddots & \vdots \\ C_{n,1} & \cdots & C_{n,p} \end{pmatrix}.$$

So we have

$$\begin{aligned}
(AC)_{j,\cdot} &= \left(\begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & \ddots & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{pmatrix} \begin{pmatrix} C_{1,1} & \cdots & C_{1,p} \\ \vdots & \ddots & \vdots \\ C_{n,1} & \cdots & C_{n,p} \end{pmatrix} \right)_{j,\cdot} \\
&= \begin{pmatrix} \sum_{k=1}^{n} A_{1,k}C_{k,1} & \cdots & \sum_{k=1}^{n} A_{1,k}C_{k,p} \\ \vdots & \ddots & \vdots \\ \sum_{k=1}^{n} A_{m,k}C_{k,1} & \cdots & \sum_{k=1}^{n} A_{m,k}C_{k,p} \end{pmatrix}_{j,\cdot} \\
&= \begin{pmatrix} \sum_{k=1}^{n} A_{j,k}C_{k,1} & \cdots & \sum_{k=1}^{n} A_{j,k}C_{k,p} \end{pmatrix} \\
&= \begin{pmatrix} A_{j,1} & \cdots & A_{j,n} \end{pmatrix} \begin{pmatrix} C_{1,1} & \cdots & C_{1,p} \\ \vdots & \ddots & \vdots \\ C_{n,1} & \cdots & C_{n,p} \end{pmatrix} \\
&= A_{j,\cdot}C,
\end{aligned}$$

as desired. □

3.C.11. Suppose $a = \begin{pmatrix} a_1 & \cdots & a_n \end{pmatrix}$ is a $1 \times n$ matrix and C is an $n \times p$ matrix. Prove that

$$aC = a_1 C_{1,\cdot} + \cdots + a_n C_{n,\cdot}.$$

In other words, show that aC is a linear combination of the rows of C, with the scalars that multiply the rows coming from a.

Proof. Let n, p be positive integers, and suppose we have $C_{j,k} \in \mathbb{F}$ for any $j = 1, \ldots, n$ and for any $k = 1, \ldots, p$. Since C is an $n \times p$ matrix, we can write

$$C = \begin{pmatrix} C_{1,1} & \cdots & C_{1,p} \\ \vdots & \ddots & \vdots \\ C_{n,1} & \cdots & C_{n,p} \end{pmatrix}.$$

So we have

$$\begin{aligned}
aC &= \begin{pmatrix} a_1 & \cdots & a_n \end{pmatrix} \begin{pmatrix} C_{1,1} & \cdots & C_{1,p} \\ \vdots & \ddots & \vdots \\ C_{n,1} & \cdots & C_{n,p} \end{pmatrix} \\
&= \begin{pmatrix} \sum_{k=1}^{n} a_k C_{k,1} & \cdots & \sum_{k=1}^{n} a_k C_{k,p} \end{pmatrix} \\
&= \begin{pmatrix} a_1 C_{1,1} + \cdots + a_n C_{n,1} & \cdots & a_1 C_{1,p} + \cdots + a_n C_{n,p} \end{pmatrix} \\
&= \begin{pmatrix} a_1 C_{1,1} & \cdots & a_1 C_{1,p} \end{pmatrix} + \cdots + \begin{pmatrix} a_n C_{n,1} & \cdots & a_n C_{n,p} \end{pmatrix} \\
&= a_1 \begin{pmatrix} C_{1,1} & \cdots & C_{1,p} \end{pmatrix} + \cdots + a_n \begin{pmatrix} C_{n,1} & \cdots & C_{n,p} \end{pmatrix} \\
&= a_1 C_{1,\cdot} + \cdots + a_n C_{n,\cdot},
\end{aligned}$$

as desired. □

3.C.13. Prove that the distributive property holds for matrix addition and matrix multiplication. In other words, suppose A, B, C, D, E, F are matrices whose sizes are such that $A(B + C)$ and $(D + E)F$ make sense. Prove that $AB + AC$ and $DF + EF$ both make sense and that $A(B + C) = AB + AC$ and $(D + E)F = DF + EF$.

Proof. Since we assumed that $A(B + C)$ makes sense, the number of rows of A equals the number of columns of $B + C$, and B and C must both have the same size. Let $A_{ij} \in \mathbb{F}$, for each $i = 1, \ldots, m$ and for each $j = 1, \ldots, n$, be entries of the $m \times n$ matrix

$$A = \begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & \ddots & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{pmatrix},$$

and let $B_{jk}, C_{jk} \in \mathbb{F}$, for each $j = 1, \ldots, n$ and for each $k = 1, \ldots, p$, be entries of the $n \times p$ matrices

$$B = \begin{pmatrix} B_{1,1} & \cdots & B_{1,p} \\ \vdots & \ddots & \vdots \\ B_{n,1} & \cdots & B_{n,p} \end{pmatrix}$$

and

$$C = \begin{pmatrix} C_{1,1} & \cdots & C_{1,p} \\ \vdots & \ddots & \vdots \\ C_{n,1} & \cdots & C_{n,p} \end{pmatrix}.$$

So AB and AC are both $m \times p$ matrices, which means $AB + AC$ makes sense. Since we assumed that $(D + E)F$ makes sense, the number of rows of $D + E$ equals the number of columns of F, and D and E must both have the same size. Let $D_{ij}, E_{ij} \in \mathbb{F}$, for each $i = 1, \ldots, m$ and for each $j = 1, \ldots, n$, be entries of the $m \times n$ matrices

$$D = \begin{pmatrix} D_{1,1} & \cdots & D_{1,n} \\ \vdots & \ddots & \vdots \\ D_{m,1} & \cdots & D_{m,n} \end{pmatrix}$$

and

$$E = \begin{pmatrix} E_{1,1} & \cdots & E_{1,n} \\ \vdots & \ddots & \vdots \\ E_{m,1} & \cdots & E_{m,n} \end{pmatrix},$$

and let $F_{jk} \in \mathbb{F}$, for each $j = 1, \ldots, n$ and for each $k = 1, \ldots, n$, be entries of the $n \times p$ matrix

$$F = \begin{pmatrix} F_{1,1} & \cdots & F_{1,p} \\ \vdots & \ddots & \vdots \\ F_{n,1} & \cdots & F_{n,p} \end{pmatrix}.$$

So DF and EF are both $m \times p$ matrices, which means $DF + EF$ makes sense. And we have

$$\begin{aligned}
A(B+C) &= \begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & \ddots & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{pmatrix} \left(\begin{pmatrix} B_{1,1} & \cdots & B_{1,p} \\ \vdots & \ddots & \vdots \\ B_{n,1} & \cdots & B_{n,p} \end{pmatrix} + \begin{pmatrix} C_{1,1} & \cdots & C_{1,p} \\ \vdots & \ddots & \vdots \\ C_{n,1} & \cdots & C_{n,p} \end{pmatrix} \right) \\
&= \begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & \ddots & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{pmatrix} \begin{pmatrix} B_{1,1} + C_{1,1} & \cdots & B_{1,p} + C_{1,p} \\ \vdots & \ddots & \vdots \\ B_{n,1} + C_{n,1} & \cdots & B_{n,p} + C_{n,p} \end{pmatrix} \\
&= \begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & \ddots & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{pmatrix} \begin{pmatrix} (B+C)_{1,1} & \cdots & (B+C)_{1,p} \\ \vdots & \ddots & \vdots \\ (B+C)_{n,1} & \cdots & (B+C)_{n,p} \end{pmatrix} \\
&= \begin{pmatrix} \sum_{r=1}^{n} A_{1,r}(B+C)_{r,1} & \cdots & \sum_{r=1}^{n} A_{1,r}(B+C)_{r,p} \\ \vdots & \ddots & \vdots \\ \sum_{r=1}^{n} A_{n,r}(B+C)_{r,1} & \cdots & \sum_{r=1}^{n} A_{n,r}(B+C)_{r,p} \end{pmatrix} \\
&= \begin{pmatrix} \sum_{r=1}^{n} A_{1,r}(B_{r,1} + C_{r,1}) & \cdots & \sum_{r=1}^{n} A_{1,r}(B_{r,p} + C_{r,p}) \\ \vdots & \ddots & \vdots \\ \sum_{r=1}^{n} A_{n,r}(B_{r,1} + C_{r,1}) & \cdots & \sum_{r=1}^{n} A_{n,r}(B_{r,p} + C_{r,p}) \end{pmatrix} \\
&= \begin{pmatrix} \sum_{r=1}^{n} A_{1,r}B_{r,1} + \sum_{r=1}^{n} A_{1,r}C_{r,1} & \cdots & \sum_{r=1}^{n} A_{1,r}B_{r,p} + \sum_{r=1}^{n} A_{1,r}C_{r,p} \\ \vdots & \ddots & \vdots \\ \sum_{r=1}^{n} A_{n,r}B_{r,1} + \sum_{r=1}^{n} A_{n,r}C_{r,1} & \cdots & \sum_{r=1}^{n} A_{n,r}B_{r,p} + \sum_{r=1}^{n} A_{n,r}C_{r,p} \end{pmatrix} \\
&= \begin{pmatrix} \sum_{r=1}^{n} A_{1,r}B_{r,1} & \cdots & \sum_{r=1}^{n} A_{1,r}B_{r,p} \\ \vdots & \ddots & \vdots \\ \sum_{r=1}^{n} A_{n,r}B_{r,1} & \cdots & \sum_{r=1}^{n} A_{n,r}B_{r,p} \end{pmatrix} + \begin{pmatrix} \sum_{r=1}^{n} A_{1,r}C_{r,1} & \cdots & \sum_{r=1}^{n} A_{1,r}C_{r,p} \\ \vdots & \ddots & \vdots \\ \sum_{r=1}^{n} A_{n,r}C_{r,1} & \cdots & \sum_{r=1}^{n} A_{n,r}C_{r,p} \end{pmatrix} \\
&= AB + AC
\end{aligned}$$

and

$$
\begin{aligned}
(D+E)F &= \left(\begin{pmatrix} D_{1,1} & \cdots & D_{1,n} \\ \vdots & \ddots & \vdots \\ D_{m,1} & \cdots & D_{m,n} \end{pmatrix} + \begin{pmatrix} E_{1,1} & \cdots & E_{1,n} \\ \vdots & \ddots & \vdots \\ E_{m,1} & \cdots & E_{n,m} \end{pmatrix}\right)\begin{pmatrix} F_{1,1} & \cdots & F_{1,p} \\ \vdots & \ddots & \vdots \\ F_{m,1} & \cdots & F_{n,p} \end{pmatrix} \\
&= \begin{pmatrix} D_{1,1}+E_{1,1} & \cdots & D_{1,p}+E_{1,p} \\ \vdots & \ddots & \vdots \\ D_{m,1}+E_{m,1} & \cdots & D_{m,p}+E_{m,p} \end{pmatrix}\begin{pmatrix} F_{1,1} & \cdots & F_{1,p} \\ \vdots & \ddots & \vdots \\ F_{n,1} & \cdots & F_{n,p} \end{pmatrix} \\
&= \begin{pmatrix} (D+E)_{1,1} & \cdots & (D+E)_{1,n} \\ \vdots & \ddots & \vdots \\ (D+E)_{m,1} & \cdots & (D+E)_{m,n} \end{pmatrix}\begin{pmatrix} F_{1,1} & \cdots & F_{1,p} \\ \vdots & \ddots & \vdots \\ F_{n,1} & \cdots & F_{n,p} \end{pmatrix} \\
&= \begin{pmatrix} \sum_{r=1}^{n}(D+E)_{1,r}F_{r,1} & \cdots & \sum_{r=1}^{n}(D+E)_{1,r}F_{r,p} \\ \vdots & \ddots & \vdots \\ \sum_{r=1}^{n}(D+E)_{m,r}F_{r,1} & \cdots & \sum_{r=1}^{n}(D+E)_{m,r}F_{r,p} \end{pmatrix} \\
&= \begin{pmatrix} \sum_{r=1}^{n}(D_{1,r}+E_{1,r})F_{r,1} & \cdots & \sum_{r=1}^{n}(D_{1,r}+E_{1,r})F_{r,p} \\ \vdots & \ddots & \vdots \\ \sum_{r=1}^{n}(D_{m,r}+E_{m,r})F_{r,1} & \cdots & \sum_{r=1}^{n}(D_{m,r}+E_{m,r})F_{r,p} \end{pmatrix} \\
&= \begin{pmatrix} \sum_{r=1}^{n}D_{1,r}F_{r,1}+\sum_{r=1}^{n}E_{1,r}F_{r,1} & \cdots & \sum_{r=1}^{n}D_{1,r}F_{r,p}+\sum_{r=1}^{n}E_{1,r}F_{r,p} \\ \vdots & \ddots & \vdots \\ \sum_{r=1}^{n}D_{m,r}F_{r,1}+\sum_{r=1}^{n}E_{m,r}F_{r,1} & \cdots & \sum_{r=1}^{n}D_{m,r}F_{r,p}+\sum_{r=1}^{n}E_{m,r}F_{r,p} \end{pmatrix} \\
&= \begin{pmatrix} \sum_{r=1}^{n}D_{1,r}F_{r,1} & \cdots & \sum_{r=1}^{n}D_{1,r}F_{r,p} \\ \vdots & \ddots & \vdots \\ \sum_{r=1}^{n}D_{m,r}F_{r,1} & \cdots & \sum_{r=1}^{n}D_{m,r}F_{r,p} \end{pmatrix} + \begin{pmatrix} \sum_{r=1}^{n}E_{1,r}F_{r,1} & \cdots & \sum_{r=1}^{n}E_{1,r}F_{r,p} \\ \vdots & \ddots & \vdots \\ \sum_{r=1}^{n}E_{m,r}F_{r,1} & \cdots & \sum_{r=1}^{n}E_{m,r}F_{r,p} \end{pmatrix} \\
&= DF + EF,
\end{aligned}
$$

as desired. □

3.C.14. Prove that matrix multiplication is associative. In other words, suppose A, B, C are matrices whose sizes are such that $(AB)C$ makes sense. Prove that $A(BC)$ makes sense and that $(AB)C = A(BC)$.

Proof. Since we assumed that $(AB)C$ makes sense, the number of rows of AB equals the number of columns of C, and A must have the same number of rows as the number of columns of B. Let $A_{ij} \in \mathbb{F}$, for each $i = 1, \ldots, m$ and for each $j = 1, \ldots, n$, be entries of the $m \times n$ matrix

$$A = \begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & \ddots & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{pmatrix},$$

let $B_{jk} \in \mathbb{F}$, for each $j = 1, \ldots, n$ and for each $k = 1, \ldots, p$, be entries of the $n \times p$ matrix

$$B = \begin{pmatrix} B_{1,1} & \cdots & B_{1,p} \\ \vdots & \ddots & \vdots \\ B_{n,1} & \cdots & B_{n,p} \end{pmatrix}$$

and let $C_{kl} \in \mathbb{F}$, for each $k = 1, \ldots, p$ and for each $l = 1, \ldots, q$, be entries of the $p \times q$ matrix

$$C = \begin{pmatrix} C_{1,1} & \cdots & C_{1,q} \\ \vdots & \ddots & \vdots \\ C_{p,1} & \cdots & C_{p,q} \end{pmatrix}.$$

Then BC is an $n \times q$ matrix, and in turn $A(BC)$ is an $m \times q$ matrix, which means $A(BC)$ makes sense. And we have

$$\begin{aligned}
(AB)C &= \left(\begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & \ddots & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{pmatrix}\begin{pmatrix} B_{1,1} & \cdots & B_{1,p} \\ \vdots & \ddots & \vdots \\ B_{n,1} & \cdots & B_{n,p} \end{pmatrix}\right)\begin{pmatrix} C_{1,1} & \cdots & C_{1,q} \\ \vdots & \ddots & \vdots \\ C_{p,1} & \cdots & C_{p,q} \end{pmatrix} \\
&= \begin{pmatrix} \sum_{r=1}^{n} A_{1,r}B_{r,1} & \cdots & \sum_{r=1}^{n} A_{1,r}B_{r,p} \\ \vdots & \ddots & \vdots \\ \sum_{r=1}^{n} A_{m,r}B_{r,1} & \cdots & \sum_{r=1}^{n} A_{m,r}B_{r,p} \end{pmatrix}\begin{pmatrix} C_{1,1} & \cdots & C_{1,q} \\ \vdots & \ddots & \vdots \\ C_{p,1} & \cdots & C_{p,q} \end{pmatrix} \\
&= \begin{pmatrix} \sum_{s=1}^{p}\left(\sum_{r=1}^{n} A_{1,r}B_{r,s}\right)C_{s,1} & \cdots & \sum_{s=1}^{p}\left(\sum_{r=1}^{n} A_{1,r}B_{r,s}\right)C_{s,q} \\ \vdots & \ddots & \vdots \\ \sum_{s=1}^{p}\left(\sum_{r=1}^{n} A_{m,r}B_{r,s}\right)C_{s,1} & \cdots & \sum_{s=1}^{p}\left(\sum_{r=1}^{n} A_{m,r}B_{r,s}\right)C_{s,q} \end{pmatrix} \\
&= \begin{pmatrix} \sum_{r=1}^{n} A_{1,r}\left(\sum_{s=1}^{p} B_{r,s}C_{s,1}\right) & \cdots & \sum_{r=1}^{n} A_{1,r}\left(\sum_{s=1}^{p} B_{r,s}C_{s,q}\right) \\ \vdots & \ddots & \vdots \\ \sum_{r=1}^{n} A_{m,r}\left(\sum_{s=1}^{p} B_{r,s}C_{s,1}\right) & \cdots & \sum_{r=1}^{n} A_{m,r}\left(\sum_{s=1}^{p} B_{r,s}C_{s,q}\right) \end{pmatrix} \\
&= \begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & \ddots & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{pmatrix}\begin{pmatrix} \sum_{s=1}^{p} B_{1,s}C_{s,1} & \cdots & \sum_{s=1}^{p} B_{1,s}C_{s,q} \\ \vdots & \ddots & \vdots \\ \sum_{s=1}^{p} B_{n,s}C_{s,1} & \cdots & \sum_{s=1}^{p} B_{n,s}C_{s,q} \end{pmatrix} \\
&= \begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & \ddots & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{pmatrix}\left(\begin{pmatrix} B_{1,1} & \cdots & B_{1,p} \\ \vdots & \ddots & \vdots \\ B_{n,1} & \cdots & B_{n,p} \end{pmatrix}\begin{pmatrix} C_{1,1} & \cdots & C_{1,q} \\ \vdots & \ddots & \vdots \\ C_{p,1} & \cdots & C_{p,q} \end{pmatrix}\right) \\
&= A(BC),
\end{aligned}$$

as desired. □

3.C.15. Suppose A is an $n \times n$ matrix and $j, k = 1, \ldots, n$. Show that the entry in row j, colum k, of A^3 (which is defined to mean AAA) is

$$(A^3)_{j,k} = \sum_{p=1}^{n}\sum_{r=1}^{n} A_{j,p}A_{p,r}A_{r,k}.$$

Proof. Let n be a positive integer, and suppose we have $A_{j,k} \in \mathbb{F}$ for any $j, k = 1, \ldots, n$. Since A is an $n \times n$ matrix, we can write

$$A = \begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & \ddots & \vdots \\ A_{n,1} & \cdots & A_{n,n} \end{pmatrix}.$$

Now, for our ease of notation, define the $n \times n$ matrix $B = A^2$ and define

$$B_{j,k} = \sum_{p=1}^{n} A_{j,p}A_{p,k}$$

for any $j, k = 1, \ldots, n$. Then we have $B_{j,k} \in \mathbb{F}$ for any $j, k = 1, \ldots, n$, and so we have

$$\begin{aligned}
B &= A^2 \\
&= AA \\
&= \begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & \ddots & \vdots \\ A_{n,1} & \cdots & A_{n,n} \end{pmatrix} \begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & \ddots & \vdots \\ A_{n,1} & \cdots & A_{n,n} \end{pmatrix} \\
&= \begin{pmatrix} \sum_{p=1}^{n} A_{1,p}A_{p,1} & \cdots & \sum_{p=1}^{n} A_{1,p}A_{p,n} \\ \vdots & \ddots & \vdots \\ \sum_{p=1}^{n} A_{n,p}A_{p,1} & \cdots & \sum_{p=1}^{n} A_{n,p}A_{p,n} \end{pmatrix} \\
&= \begin{pmatrix} B_{1,1} & \cdots & B_{1,n} \\ \vdots & \ddots & \vdots \\ B_{n,1} & \cdots & B_{n,n} \end{pmatrix}.
\end{aligned}$$

In other words, for all $j, k = 1, \ldots, n$, the element $B_{j,k} \in \mathbb{F}$ denotes the entry in row j, column k of B. So we have

$$\begin{aligned}
A^3 &= A^2 A \\
&= BA \\
&= \begin{pmatrix} B_{1,1} & \cdots & B_{1,n} \\ \vdots & \ddots & \vdots \\ B_{n,1} & \cdots & B_{n,n} \end{pmatrix} \begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & \ddots & \vdots \\ A_{n,1} & \cdots & A_{n,n} \end{pmatrix} \\
&= \begin{pmatrix} \sum_{r=1}^{n} B_{1,r}A_{r,1} & \cdots & \sum_{r=1}^{n} B_{1,r}A_{r,n} \\ \vdots & \ddots & \vdots \\ \sum_{r=1}^{n} B_{n,r}A_{r,1} & \cdots & \sum_{r=1}^{n} B_{n,r}A_{r,n} \end{pmatrix}.
\end{aligned}$$

Therefore, the entry in row j, column k of A^3 is

$$\begin{aligned}
(A^3)_{j,k} &= \sum_{r=1}^{n} B_{j,r}A_{r,k} \\
&= \sum_{r=1}^{n} \left(\sum_{p=1}^{n} A_{j,p}A_{p,r} \right) A_{r,k} \\
&= \sum_{p=1}^{n} \sum_{r=1}^{n} A_{j,p}A_{p,r}A_{r,k},
\end{aligned}$$

as desired. □

3.D.1. Suppose $T \in \mathcal{L}(U, V)$ and $S \in \mathcal{L}(V, W)$ are both invertible linear maps. Prove that $ST \in \mathcal{L}(U, W)$ is invertible and that $(ST)^{-1} = T^{-1}S^{-1}$.

Proof. Since S and T are both invertible, their respective inverses S^{-1} and T^{-1} exist. Let I_V and I_W be identity maps on V and W, respectively. We have

$$\begin{aligned}
(T^{-1}S^{-1})(ST) &= T^{-1}(S^{-1}S)T \\
&= T^{-1}I_V T \\
&= T^{-1}T \\
&= I_U
\end{aligned}$$

and

$$\begin{aligned}
(ST)(T^{-1}S^{-1}) &= S(TT^{-1})S^{-1} \\
&= SI_U S^{-1} \\
&= SS^{-1} \\
&= I_V.
\end{aligned}$$

Therefore, ST is invertible with inverse $(ST)^{-1} = T^{-1}S^{-1}$. □

3.D.9. Suppose V is finite-dimensional and $S, T \in \mathcal{L}(V)$. Prove that ST is invertible if and only if both S and T are invertible.

Proof. Forward direction: If ST is invertible, then both S and T are invertible. Since ST is invertible, there exists $R \in \mathcal{L}(V)$ that satisfies $R(ST) = I$ and $(ST)R = I$, where I is the identity map on V. First, we will prove that T is injective. Suppose we have $v \in \operatorname{null} T$, meaning that $v \in V$ satisfies $Tv = 0$. Then we have

$$\begin{aligned} v &= Iv \\ &= (R(ST))v \\ &= RS(Tv) \\ &= RS(0) \\ &= R(0) \\ &= 0. \end{aligned}$$

Therefore, we have $v \in \{0\}$, and so $\operatorname{null} T \subset \{0\}$. By 3.14 of Axler, $\operatorname{null} T$ is a subspace of V, so in particular we have $0 \in \operatorname{null} T$, or $\{0\} \subset \operatorname{null} T$. So we conclude the set equality $\operatorname{null} T = \{0\}$. We conclude by 3.16 of Axler that T is injective. Since V is finite-dimensional and $T \in \mathcal{L}(V)$, saying T is injective is equivalent to saying that T is invertible, according to 3.69 of Axler.

Backward direction: If S and T are both invertible, then ST is invertible. We already proved this in Exercise 3.D.1. □

3.D.10. Suppose V is finite-dimensional and $S, T \in \mathcal{L}(V)$. Prove that $ST = I$ if and only if $TS = I$.

Proof. Forward direction: If $ST = I$, then $TS = I$. Suppose we have $ST = I$. As an identity map, I is automatically invertible, which means ST is invertible. By Exercise 3.D.9 of Axler, S and T are both invertible, which means their respective inverses S^{-1} and T^{-1} exist. So we obtain

$$\begin{aligned} TS &= TS(TT^{-1}) \\ &= T(ST)T^{-1} \\ &= TIT^{-1} \\ &= TT^{-1} \\ &= I, \end{aligned}$$

as desired.

Backward direction: If $TS = I$, then $ST = I$. We can interchange the roles of S and T from the last proof. Suppose we have $TS = I$. As an identity map, I is automatically invertible, which means TS is invertible. By Exercise 3.D.9 of Axler, S and T are both invertible, which means their respective inverses S^{-1} and T^{-1} exist. So we obtain

$$\begin{aligned} ST &= ST(SS^{-1}) \\ &= S(TS)S^{-1} \\ &= SIS^{-1} \\ &= SS^{-1} \\ &= I, \end{aligned}$$

as desired. □

3.D.11. Suppose V is finite-dimensional, $S, T, U \in \mathcal{L}(V)$, and $STU = I$. Show that T is invertible and that $T^{-1} = US$.

Proof. Since we have $STU = I$ and, as an identity map, I is invertible, it follows by Exercise 3.D.9 of Axler that S and TU are invertible. Since TU is invertible, it follows by Exercise 3.D.9 of Axler again that T and U are invertible. In particular, T is invertible. There exist maps S^{-1}, T^{-1}, U^{-1}, which are respective inverses of S, T, U. So we have

$$\begin{aligned} US &= IUS \\ &= (T^{-1}T)US \\ &= T^{-1}TUS \\ &= T^{-1}ITUS \\ &= T^{-1}(S^{-1}S)TUS \\ &= T^{-1}S^{-1}(STU)S \\ &= T^{-1}S^{-1}IS \\ &= T^{-1}S^{-1}S \\ &= T^{-1}I \\ &= T^{-1}, \end{aligned}$$

as desired. □

3.D.13. Suppose V is a finite-dimensional vector space and $R, S, T \in \mathcal{L}(V)$ are such that RST is surjective. Prove that S is injective.

Proof. Since V is finite-dimensional and RST is surjective, 3.69 of Axler states that RST is also invertible. By Exercise 3.D.9 of Axler, RS is invertible and T is invertible. Since RS is invertible, by Exercise 3.E.9 of Axler again, R and S is invertible. In particular, S is invertible. By 3.69 of Axler, S is injective. □

3.D.14. Suppose $v_1, \ldots, v_n$ is a basis of V. Prove that the map $T : V \longrightarrow \mathbb{F}^{n,1}$ defined by

$$Tv = \mathcal{M}(v)$$

is an isomorphism of V onto $\mathbb{F}^{n,1}$; here $\mathcal{M}(v)$ is the matrix of $v \in V$ with respect to the basis $v_1, \ldots, v_n$.

Proof. To show that $T : V \longrightarrow \mathbb{F}^{n,1}$ is an isomorphism, we need to show that T is linear and invertible. First, we will show that T is linear. Since $v_1, \ldots, v_n$ is a basis of V, there exist $a_1, \ldots, a_n, b_1, \ldots, b_n \in \mathbb{F}$ such that

$$u = a_1 v_1 + \cdots + a_n v_n$$

and

$$v = b_1 v_1 + \cdots + b_n v_n.$$

So, for all $u, v \in V$ and for all $\lambda \in \mathbb{F}$, we have

$$\begin{aligned} T(u+v) &= \mathcal{M}(u+v) \\ &= \mathcal{M}((a_1 v_1 + \cdots + a_n v_n) + (b_1 v_1 + \cdots + b_n v_n)) \\ &= \mathcal{M}((a_1 + b_1) v_1 + \cdots + (a_n + b_n) v_n) \\ &= \begin{pmatrix} a_1 + b_1 \\ \vdots \\ a_n + b_n \end{pmatrix} \\ &= \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} + \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \\ &= \mathcal{M}(a_1 v_1 + \cdots + a_n v_n) + \mathcal{M}(b_1 v_1 + \cdots + b_n v_n) \\ &= \mathcal{M}(u) + \mathcal{M}(v) \\ &= Tu + Tv, \end{aligned}$$

satisfying additivity, and

$$\begin{aligned} T(\lambda u) &= \mathcal{M}(\lambda u) \\ &= \mathcal{M}(\lambda(a_1 v_1 + \cdots + a_n v_n)) \\ &= \mathcal{M}((\lambda a_1) v_1 + \cdots + (\lambda a_n) v_n) \\ &= \begin{pmatrix} \lambda a_1 \\ \vdots \\ \lambda a_n \end{pmatrix} \\ &= \lambda \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \\ &= \lambda \mathcal{M}(a_1 v_1 + \cdots + a_n v_n) \\ &= \lambda \mathcal{M}(u) \\ &= \lambda T u, \end{aligned}$$

satisfying homogeneity. So T is linear. Next, we will show that T is invertible. According to 3.56 of Axler, this is equivalent to showing that T is injective and surjective. First, we will show that T is injective. Suppose $Tu = 0$. Since we have from earlier $u = a_1 v_1 + \cdots + a_n v_n$, we get

$$\begin{aligned} \mathcal{M}(a_1 v_1 + \cdots + a_n v_n) &= \mathcal{M}(u) \\ &= Tu \\ &= 0. \end{aligned}$$

We can write both sides of the above equation as matrices:

$$\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

So our scalars are $a_1 = 0, \ldots, a_n = 0$, which means

$$\begin{aligned} u &= a_1 v_1 + \cdots + a_n v_n \\ &= 0v_1 + \cdots + 0v_n \\ &= 0. \end{aligned}$$

So $\operatorname{null} T = \{0\}$, which means, by 3.16 of Axler, T is injective. Next, we will show that T is surjective. We have

$$\begin{aligned} Tu &= \mathcal{M}(u) \\ &= \mathcal{M}(a_1 v_1 + \cdots + a_n v_n) \\ &= \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}. \end{aligned}$$

Since $a_1, \ldots, a_n \in \mathbb{F}$ are arbitrary values, we conclude that T is also surjective. Therefore, T is both injective and surjective, which means T is invertible. Moreover, T is both linear and invertible, which means T is an isomorphism. □

3.D.16. Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Prove that T is a scalar multiple of the identity if and only if $ST = TS$ for all $S \in \mathcal{L}(V)$.

Proof. Forward direction: If T is a sclar multiple of the identity, then $ST = TS$ for all $S \in \mathcal{L}(V)$. Suppose T is the scalar multiple of the identity map on V. Then there exists $\lambda \in \mathbb{F}$ such that we have $T = \lambda I_V$, where I_V is the identity map on V. Therefore, for all $S \in \mathcal{L}(V)$, we get

$$\begin{aligned} ST &= S(\lambda I_V) \\ &= \lambda S I_V \\ &= \lambda S \\ &= (\lambda I_V)S \\ &= TS, \end{aligned}$$

as desired.

Backward direction: If $ST = TS$ for all $S \in \mathcal{L}(V)$, then T is a sclar multiple of the identity. Suppose that we have $ST = TS$ for all $S \in \mathcal{L}(V)$. First, we will show that, for all $v \in V$, the list v, Tv is linearly dependent. Suppose instead by contradiction that v, Tv is linearly independent. Then, according to 2.33 of Axler, we can extend v, Tv to a basis $v, Tv, u_1, \ldots, u_n$ of V. (This means the dimension of V is $\dim V = n + 2$, but that is not really important in this proof.) So every vector in V can be written in the form $av + bTv + c_1 u_1 + \cdots + c_n u_n$ for some $a, b, c_1, \ldots, c_n \in \mathbb{F}$. This means that we can define $S \in \mathcal{L}(V)$ by

$$S(av + bTv + c_1 u_1 + \cdots + c_n u_n) = bv,$$

which satisfies in particular $S(Tv) = v$ and $Sv = 0$. Therefore, since $ST = TS$, we obtain

$$\begin{aligned} v &= S(Tv) \\ &= (ST)v \\ &= (TS)v \\ &= T(Sv) \\ &= T(0) \\ &= 0, \end{aligned}$$

using 3.11 of Axler to justify the last equality above. So we can choose nonzero scalars such as $a_1 = 1, a_2 = 1 \in \mathbb{F}$ to satisfy

$$\begin{aligned} a_1 v + a_2 Tv &= 1(0) + 1T(0) \\ &= 1(0) + 1(0) \\ &= 0 + 0 \\ &= 0, \end{aligned}$$

meaning that the list v, Tv is linearly dependent. But this contradicts our assumption at the beginning that v, Tv is linearly independent. Therefore, the list v, Tv must be linearly dependent. By the Linear Dependence Lemma (2.21 of Axler), we have $Tv \in \operatorname{span}(v)$. In other words, for all nonzero $v \in V$, there exists $\lambda_v \in \mathbb{F}$ (the subscript notation signifies that the scalar λ_v depends on our choice of some vector v) such that $Tv = \lambda_v v$, which means $T = \lambda_v I_V$, where once again I_V is the identity map on V. At this stage, we have almost completed our proof. To show that T is a scalar multiple of the identity, we need to establish $T = \lambda I_V$, where $\lambda \in \mathbb{F}$ does not depend on v. In other words, it is not enough to stop at $Tv = \lambda_v v$; we need to show that λ_v is actually constant in v, at which point would allow us to write $\lambda_v = \lambda$. Let $w \in V$ be another arbitrary vector. Then

v, w form a list that is either linearly independent or linearly dependent. Consider $\lambda_v, \lambda_w, \lambda_{v+w} \in \mathbb{F}$, the scalars that depend on $v, w, v+w$, respectively. In the first case, assume that v, w is linearly independent. Applying $Tv = \lambda_v v$, we obtain

$$\begin{aligned}(\lambda_{v+w} - \lambda_v)v + (\lambda_{v+w} - \lambda_w)w &= \lambda_{v+w}v - \lambda_v v + \lambda_{v+w}w - \lambda_w w \\ &= \lambda_{v+w}(v+w) - \lambda_v v - \lambda_w w \\ &= T(v+w) - \lambda_v v - \lambda_w w \\ &= Tv + Tw - \lambda_v v - \lambda_w w \\ &= \lambda_v v + \lambda_w w - \lambda_v v - \lambda_w w \\ &= 0.\end{aligned}$$

Since v, w is linearly independent, all scalars are zero; that is, we have

$$\lambda_{v+w} - \lambda_v = 0, \lambda_{v+w} - \lambda_w = 0,$$

or $\lambda_v = \lambda_{v+w} = \lambda_w$. Any function that outputs the same value such as $\lambda_v = \lambda_w$ for all input values such as $v, w \in V$ must be a constant function; in other words, we conclude that λ_v is constant, or $\lambda_v = \lambda$. Therefore, we conclude $T = \lambda_v I_V = \lambda I_V$, which means that T is a scalar multiple of the identity. □

3.D.20. Suppose n is a positive integer and $a_{i,j} \in \mathbb{F}$ for all $i, j = 1, \ldots, n$. You may assume without proof that, since $\mathbb{F}^n$ is a finite-dimensional vector space, $T \in \mathcal{L}(\mathbb{F}^n)$ is injective if and only if T is surjective (from 3.69 of Axler). Prove that the following are equivalent (note that in both parts below, the number of equations equals the number of variables):

(a) The trivial solution $x_1 = \cdots = x_n = 0$ is the only solution to the homogeneous system of equations

$$\begin{gathered}\sum_{k=1}^{n} a_{1,k}x_k = 0 \\ \vdots \\ \sum_{k=1}^{n} a_{n,k}x_k = 0.\end{gathered}$$

(b) For all $c_1, \ldots, c_n \in \mathbb{F}$, there exists a solution to the system of equations

$$\begin{gathered}\sum_{k=1}^{n} a_{1,k}x_k = c_1 \\ \vdots \\ \sum_{k=1}^{n} a_{n,k}x_k = c_n.\end{gathered}$$

Proof. Define $T \in \mathcal{L}(\mathbb{F}^n)$ by

$$T(x_1, \ldots, x_n) = \left(\sum_{k=1}^{n} a_{1,k}x_k, \ldots, \sum_{k=1}^{n} a_{n,k}x_k\right).$$

Statement (a) says that we have

$$T(x_1, \ldots, x_n) = (0, \ldots, 0)$$

if and only if $(x_1, \ldots, x_n) = (0, \ldots, 0)$, if and only if $\operatorname{null} T = \{(0, \ldots, 0)\}$, if and only if T is injective. Statement (b) says that, for all $c_1, \ldots, c_n \in \mathbb{F}$, we have

$$T(x_1, \ldots, x_n) = (c_1, \ldots, c_n)$$

if and only if T is surjective. Since $\mathbb{F}^n$ is finite-dimensional, T is injective if and only if T is surjective, according to 3.69 of Axler. Therefore, statement (a) holds if and only if statement (b) holds. □