MATH 131: Linear Algebra I
University of California, Riverside
Homework 5 Solutions
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Solutions to assigned homework problems from *Linear Algebra Done Right* (third edition) by Sheldon Axler

3.E: 1, 7, 12, 13, 17, 18, 20
3.F: 3, 4, 6, 7, 8, 15, 32

3.E.1. Suppose $T : V \to W$ is a function. Then *graph* of T is the subset of $V \times W$ defined by

$$\text{graph of } T = \{(v, Tv) \in V \times W : v \in V\}.$$

Prove that T is a linear map if and only if the graph of T is a subspace of $V \times W$.

Proof. Forward direction: If T is a linear map, then the graph of T is a subspace of $V \times W$. Suppose T is linear. We will prove that the graph of T is a subspace of $V \times W$.

- Additive identity: Since T is linear, by 3.11 of Axler we have $T(0) = 0$. So we have $(0, 0) = (0, T(0)) \in$ graph of T.
- Closed under addition: Suppose we have $(u, Tu), (v, Tv) \in$ graph of T. Since T is linear, we can use its additivity to obtain

$$\begin{aligned}(u, Tu) + (v, Tv) &= (u + v, Tu + Tv)\\ &= (u + v, T(u + v)).\end{aligned}$$

  So we conclude $(u, Tu) + (v, Tv) \in$ graph of T.
- Closed under scalar multiplication: Suppose we have $\lambda \in \mathbb{F}$ and $(v, Tv) \in$ graph of T. Since T is linear, we can use its homogeneity to obtain

$$\begin{aligned}\lambda(v, Tv) &= (\lambda v, \lambda Tv)\\ &= (\lambda v, T(\lambda v)).\end{aligned}$$

  So we conclude $\lambda(v, Tv) \in$ graph of T.

Since we satisfied all the properties of a subspace, we conclude that the graph of T is a subspace of $V \times W$.

Backward direction: If the graph of T is a subspace of $V \times W$, then T is a linear map.

- Additivity: Suppose we have $(u, Tu), (v, Tv) \in$ graph of T. Since the graph of T is a subspace of $V \times W$, in particular it is closed under addition, which means we have $(u + v, Tu + Tv) = (u, Tu) + (v, Tv) \in$ graph of T. At the same time, all elements of the graph of T must take the form (v, Tv). So we actually have

$$(u + v, Tu + Tv) = (u + v, T(u + v)),$$

  from which we can equate the second coordinates to obtain $T(u + v) = Tu + Tv$, establishing the additivity of T.
- Homogeneity: Suppose we have $\lambda \in \mathbb{F}$ and $(v, Tv) \in$ graph of T.Since the graph of T is a subspace of $V \times W$, in particular it is closed under scalar multiplication, which means we have $(\lambda v, \lambda Tv) = \lambda(v, Tv) \in$ graph of T. At the same time, all elements of the graph of T must take the form (v, Tv). So we actually have

$$(\lambda v, \lambda Tv) = (\lambda v, T(\lambda v)),$$

  from which we can equate the second coordinates to obtain $T(\lambda v) = \lambda Tv$, establishing the homogeneity of T.

Since additivity and homogeneity of T are satisfied, we conclude that T is a linear map. □

3.E.7. Suppose v, x are vectors in V and U, W are subspaces of V such that $v + U = x + W$. Prove that $U = W$.

Proof. Since U, W are subspaces of V, they in particular satisfy the additive identity, meaning that we have $0 \in U$ and $0 \in W$. So we have

$$\begin{aligned}v &= v + 0\\ &\in v + U\\ &= x + W,\end{aligned}$$

and so there exists $w \in W$ that satisfies $v = x + w$, or equivalently, $x - v = w \in W$. Similarly, we have

$$\begin{aligned} x &= x + 0 \\ &\in x + W \\ &= v + U, \end{aligned}$$

and so there exists $u \in U$ that satisfies $x = v + u$, or equivalently, $x - v = u \in U$. By 3.85 of Axler, the statements $x - v \in W$ and $x - v \in U$ are equivalent to their respective statements $x + W = v + W$ and $x + U = v + U$. Therefore, we have

$$\begin{aligned} v + U &= x + W \\ &= v + W \end{aligned}$$

or

$$\begin{aligned} x + U &= v + U \\ &= x + W. \end{aligned}$$

In either case, we conclude $U = W$. □

3.E.12. Suppose U is a subspace of V such that V/U is finite-dimensional. Prove that V is isomorphic to $U \times (V/U)$.

Proof. Let $v_1 + U, \ldots, v_n + U$ be a basis of V/U. Then 2.29 of Axler asserts that, for all $v \in V$, we can write $v + U \in V/U$ uniquely in the form

$$\begin{aligned} v + U &= a_1(v_1 + U) + \cdots + a_n(v_n + U) \\ &= ((a_1 v_1) + U) + \cdots + ((a_n v_n) + U) \\ &= (a_1 v_1 + \cdots + a_n v_n) + U \end{aligned}$$

for some $a_1, \ldots, a_n \in \mathbb{F}$, which is equivalent to saying $v - (a_1 v_1 + \cdots + a_n v_n) \in U$ by 3.85 of Axler. Now, define the map $T : V \to U \times (V/U)$ by

$$Tv = (v - (a_1 v_1 + \cdots + a_n v_n), v + U).$$

We will prove that that T is an isomorphism. First, we will prove that T is linear.

- Additivity: As done with v in the problem statement, we can write $w + U \in V/U$ uniquely in the form

$$w + U = b_1(v_1 + U) + \cdots + b_n(v_n + U)$$

  for some $b_1, \ldots, b_n \in \mathbb{F}$. In fact, we have

$$\begin{aligned} w + U &= b_1(v_1 + U) + \cdots + b_n(v_n + U) \\ &= ((b_1 v_1) + U) + \cdots + ((b_n v_n) + U) \\ &= (b_1 v_1 + \cdots + b_n v_n) + U. \end{aligned}$$

  By 3.85 of Axler, we have $w - (b_1 v_1 + \cdots + b_n v_n) \in U$. Therefore, for all $v, w \in V$, we have

$$\begin{aligned} T(v + w) &= ((v + w) - ((a_1 + b_1)v_1 + \cdots + (a_n + b_n)v_n), (v + w) + U) \\ &= ((v - (a_1 v_1 + \cdots + a_n v_n)) + (w - (b_1 v_1 + \cdots + b_n v_n)), (v + U) + (w + U)) \\ &= (v - (a_1 v_1 + \cdots + a_n v_n), v + U) + (w - (b_1 v_1 + \cdots + b_n v_n), w + U) \\ &= Tv + Tw. \end{aligned}$$

- Homogeneity: For all $\lambda \in \mathbb{F}$ and for all $v \in V$, we have

$$\begin{aligned} T(\lambda v) &= ((\lambda v) - ((\lambda a_1)v_1 + \cdots + (\lambda a_n)v_n), (\lambda v) + U) \\ &= \lambda(v - \lambda(a_1 v_1 + \cdots + a_n v_n), \lambda(v + U)) \\ &= \lambda T v. \end{aligned}$$

Since additivity and homogeneity of T are satisfied, T is linear. Next, we need to prove that T is injective and surjective. We will prove first that T is injective. Suppose we have $v \in \operatorname{null} T$, meaning that v satisfies $Tv = (0, 0 + U)$. Then we have

$$\begin{aligned} (0, 0 + U) &= Tv \\ &= (v - (a_1 v_1 + \cdots + a_n v_n), \pi(v)) \\ &= (v - (a_1 v_1 + \cdots + a_n v_n), v + U) \\ &= (v - (a_1 v_1 + \cdots + a_n v_n), a_1(v_1 + U) + \cdots + a_n(v_n + U)), \end{aligned}$$

from which we can equate the coordinates to obtain

$$v - (a_1 v_1 + \cdots + a_n v_n) = 0$$

and

$$a_1(v_1 + U) + \cdots + a_n(v_n + U) = 0 + U.$$

Since $v_1 + U, \ldots, v_n + U$ is a basis of V/U, it is linearly independent in V/U, and so the second equation $a_1(v_1 + U) + \cdots + a_n(v_n + U) = 0 + U$ implies

$$a_1 = 0, \ldots, a_n = 0.$$

Furthermore, the first equation $v - (a_1 v_1 + \cdots + a_n v_n) = 0$ with $a_1 = 0, \ldots, a_n = 0$ implies $v = 0$. Therefore, we have $\operatorname{null} T \subset \{0\}$. But 3.14 of Axler says that $\operatorname{null} T$ is a subspace of V, which means in particular that we have $\{0\} \subset \operatorname{null} T$. Therefore, we obtain the set equality $\operatorname{null} T = \{0\}$. By 3.16 of Axler, T is injective. Now we will prove that T is surjective. Consider an arbitrary element $(u, v + U) \in U \times (V/U)$. Then we have $u - 0 = u \in U$, which, according to 3.85 of Axler, is equivalent to saying

$$u + U = 0 + U.$$

Consequently, we have

$$\begin{aligned}
v + U &= a_1(v_1 + U) + \cdots + a_n(v_n + U) \\
&= ((a_1 v_1) + U) + \cdots + ((a_n v_n) + U) \\
&= (a_1 v_1 + \cdots + a_n v_n) + U \\
&= (a_1 v_1 + \cdots + a_n v_n + 0) + U \\
&= ((a_1 v_1 + \cdots + a_n v_n) + U) + (0 + U) \\
&= ((a_1 v_1 + \cdots + a_n v_n) + U) + (u + U) \\
&= (a_1 v_1 + \cdots + a_n v_n + u) + U.
\end{aligned}$$

If we add and subtract $a_1 v_1 + \cdots + a_m v_m + u$ for u, then we can write

$$\begin{aligned}
u &= (a_1 v_1 + \cdots + a_n v_n) + u - (a_1 v_1 + \cdots + a_n v_n) \\
&= (a_1 v_1 + \cdots + a_n v_n + u) - (a_1 v_1 + \cdots + a_n v_n).
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
(u, v + U) &= ((a_1 v_1 + \cdots + a_n v_n + u) - (a_1 v_1 + \cdots + a_n v_n), (a_1 v_1 + \cdots + a_n v_n + u) + U) \\
&= T(a_1 v_1 + \cdots + a_n v_n + u),
\end{aligned}$$

which means we have $(u, v + U) \in \operatorname{range} T$, and so we get the set containment $U \times (V/U) \subset \operatorname{range} T$. But 3.19 of Axler states that $\operatorname{range} T$ is a subspace of $U \times (V/U)$. So we conclude the set equality

$$\operatorname{range} T = U \times (V/U),$$

which means T is surjective. So we established that T is both injective and surjective, which means by 3.46 of Axler T is invertible. Therefore, T is an invertible linear map, and so it is an isomorphism. □

3.E.13. Suppose U is a subspace of V and $v_1 + U, \ldots, v_m + U$ is a basis of V/U and $u_1, \ldots, u_n$ is a basis of U. Prove that $v_1, \ldots, v_m, u_1, \ldots, u_n$ is a basis of V. Suppose that $v_1 + U, \ldots, v_m + U$ is a basis of V/U and that $u_1, \ldots, u_n$ is a basis of U. Prove that $v_1, \ldots, v_m, u_1, \ldots, u_n$ is a basis of V.

Proof. First, we will show that the list $v_1, \ldots, v_m, u_1, \ldots, u_n$ is linearly independent in V. Suppose $a_1, \ldots, a_m, c_1, \ldots, c_n \in \mathbb{F}$ satisfy

$$a_1 v_1 + \cdots + a_m v_m + c_1 u_1 + \cdots + c_n u_n = 0.$$

Then we have

$$\begin{aligned}
(a_1 v_1 + \cdots + a_m v_m) - 0 &= a_1 v_1 + \cdots + a_m v_m \\
&= -c_1 u_1 - \cdots - c_n u_n \\
&\in U.
\end{aligned}$$

By 3.85 of Axler, we obtain

$$(a_1 v_1 + \cdots + a_n v_n) + U = 0 + U.$$

In fact, we get

$$\begin{aligned}
a_1(v_1 + U) + \cdots + a_m(v_m + U) &= (a_1 v_1 + U) + \cdots + (a_m v_m + U) \\
&= (a_1 v_1 + U) + \cdots + (a_m v_m + U) \\
&= (a_1 v_1 + \cdots + a_m v_m) + U \\
&= 0 + U.
\end{aligned}$$

Recall that $0 + U$ is the additive identity of V/U. Since $v_1 + U, \ldots, v_m + U$ is a basis of V/U, it is linearly independent in V/U, which means we must have

$$a_1 = 0, \ldots, a_m = 0.$$

Consequently, our original equation becomes

$$\begin{aligned} 0 &= a_1 v_1 + \cdots + a_m v_m + c_1 u_1 + \cdots + c_n u_n \\ &= 0 v_1 + \cdots + 0 v_m + c_1 u_1 + \cdots + c_n u_n \\ &= c_1 u_1 + \cdots + c_n u_n. \end{aligned}$$

Now, since $u_1, \ldots, u_n$ is a basis of U, it is linearly independent in U, which means we must have

$$c_1 = 0, \ldots, c_n = 0.$$

Altogether, we have

$$a_1 = 0, \ldots, a_m = 0, c_1 = 0, \ldots, c_n = 0.$$

Therefore, the list $v_1, \ldots, v_m, u_1, \ldots, u_n$ is linearly independent in V. Next, we must show that $v_1, \ldots, v_m, u_1, \ldots, u_n$ spans V. Since $v_1 + U, \ldots, v_m + U$ is a basis of V/U, it spans U, and so, for all $v \in V$, we can write every element in V/U uniquely in the form

$$v + U = a_1(v_1 + U) + \cdots + a_m(v_m + U)$$

for some $a_1, \ldots, a_m \in \mathbb{F}$. In fact, when applying the operations of addition and scalar multiplication defined on V/U, we obtain

$$\begin{aligned} v + U &= a_1(v_1 + U) + \cdots + a_m(v_m + U) \\ &= (a_1 v_1 + U) + \cdots + (a_m v_m + U) \\ &= (a_1 v_1 + \cdots + a_m v_m) + U. \end{aligned}$$

By 3.85 of Axler, we have

$$v - (a_1 v_1 + \cdots + a_m v_m) \in U.$$

Since $u_1, \ldots, u_m$ is a basis of U, it spans U, and so we can write every vector in U as a linear combination of $u_1, \ldots, u_m$. In particular, we can write

$$v - (a_1 v_1 + \cdots + a_m v_m) = c_1 u_1 + \cdots + c_n u_n$$

for some $c_1, \ldots, c_n \in \mathbb{F}$. Therefore, we have

$$\begin{aligned} v &= (a_1 v_1 + \cdots + a_m v_m) + (c_1 u_1 + \cdots + c_n u_n) \\ &= a_1 v_1 + \cdots + a_m v_m + c_1 u_1 + \cdots + c_n u_n, \end{aligned}$$

where $a_1, \ldots, a_m, c_1, \ldots, c_n \in \mathbb{F}$. Since $v \in V$ is arbitrary, we conclude that the list $v_1, \ldots, v_m, u_1, \ldots, u_n$ spans V. Therefore, $v_1, \ldots, v_m, u_1, \ldots, u_n$ is a basis of V. □

Alternate proof. We already showed in our original proof to this exercise that the list $v_1, \ldots, v_m, u_1, \ldots, u_n$ is linearly independent in V. We will show another way of proving that this list is a basis of V. By Exercise 3.E.12 of Axler, V is isomorphic to $U \times (V/U)$. By 3.59 of Axler, we have

$$\dim V = \dim(U \times (V/U)).$$

Since $u_1, \ldots, u_n$ is a basis of U and $v_1 + U, \ldots, v_m + U$ is a basis of V/U, it follows that we have $\dim U = m$ and $\dim(V/U) = n$, respectively. This means that U and V/U are both finite-dimensional, and so 3.76 of Axler gives us

$$\dim(U \times V/U) = \dim U + \dim(V/U).$$

Therefore, we have

$$\begin{aligned} \dim V &= \dim(U \times (V/U)) \\ &= \dim U + \dim(V/U) \\ &= m + n. \end{aligned}$$

We notice that our linearly independent list $v_1, \ldots, v_m, u_1, \ldots, u_n$ has length $m + n$; in other words, this linearly independent list has the right length. By 2.39 of Axler, $v_1, \ldots, v_m, u_1, \ldots, u_n$ is a basis of V. □

3.E.17. Suppose U is a subspace of V such that V/U is finite-dimensional. Prove that there exists a subspace W of V such that $\dim W = \dim V/U$ and $V = U \oplus W$.

Proof. Since V/U is finite-dimensional, by 2.37 of Axler, there exist $v_1, \ldots, v_n \in V$ such that $v_1 + U, \ldots, v_n + U$ is a basis of V/U. In other words, $v_1 + U, \ldots, v_n + U$ is a linearly independent list that spans V/U. Since the list spans V/U, every vector in V/U can be written

$$\begin{aligned} v + U &= a_1(v_1 + U) + \cdots + a_n(v_n + U) \\ &= (a_1 v_1 + U) + \cdots + (a_n v_n + U) \\ &= (a_1 v_1 + \cdots + a_n v_n) + U \end{aligned}$$

for some $a_1, \ldots, a_n \in \mathbb{F}$. By 3.85 of Axler, this is equivalent to saying $v - (a_1 v_1 + \cdots + a_n v_n) \in U$. Now let $W = \operatorname{span}(v_1, \ldots, v_n)$. By construction, the list $v_1, \ldots, v_n$ spans W and, by 2.7 of Axler, W is a subspace of V. We need to show that $v_1, \ldots, v_n$ is linearly independent in W. Suppose by contradiction that $v_1, \ldots, v_n$ is linearly dependent in W. Then there exist $b_1, \ldots, b_n \in \mathbb{F}$, not all zero, that satisfy

$$b_1 v_1 + \cdots + b_n v_n = 0.$$

So we have

$$\begin{aligned} b_1(v_1 + U) + \cdots + b_n(v_n + U) &= ((b_1 v_1) + U) + \cdots + ((b_n v_n) + U) \\ &= (b_1 v_1 + \cdots + b_n v_n) + U \\ &= 0 + U, \end{aligned}$$

which means $v_1 + U, \ldots, v_n + U$ is linearly dependent in V/U. But this contradicts our earlier result saying that $v_1 + U, \ldots, v_n + U$ is linearly independent in V/U. Therefore, $v_1, \ldots, v_n$ is linearly independent in W. So we proved that $v_1, \ldots, v_n$ is a linearly independent list that spans W, and so $v_1, \ldots, v_n$ is a basis of W. So we have $\dim W = n = \dim(V/U)$, and every vector in W can be written in the form $a_1 v_1 + \cdots + a_n v_n$ for some $a_1, \ldots, a_n \in \mathbb{F}$. Therefore, if we have $v \in V$, then we can write

$$\begin{aligned} v &= (v - (a_1 v_1 + \cdots + a_n v_n)) + (a_1 v_1 + \cdots + a_n v_n) \\ &\in U + W. \end{aligned}$$

Therefore, we have $V \subset U + W$. At the same time, by 1.39 of Axler, that $U + W$ is a subspace of V. Therefore, we have $V = U + W$. Next, we need to establish $U \cap W = \{0\}$. Suppose we have $v \in U \cap W$. Then we have $v \in U$ and $v \in W$, the latter of which means that, with the basis $v_1, \ldots, v_n$ of W, we can write

$$v = a_1 v_1 + \cdots + a_n v_n$$

for some $a_1, \ldots, a_n \in \mathbb{F}$. Since we have $v - 0 = v \in U$, by 3.85 of Axler it is equivalent to saying $v + U = 0 + U$. In fact, we have

$$\begin{aligned} 0 + U &= v + U \\ &= (a_1 v_1 + \cdots + a_n v_n) + U \\ &= ((a_1 v_1) + U) + \cdots + ((a_n v_n) + U) \\ &= a_1(v_1 + U) + \cdots + a_n(v_n + U). \end{aligned}$$

Since $v_1 + U, \ldots, v_n + U$ is a basis of V/U, it is linearly independent in V/U, which means we must have

$$a_1 = 0, \ldots, a_n = 0.$$

Therefore, we conclude

$$\begin{aligned} v &= a_1 v_1 + \cdots + a_n v_n \\ &= 0 v_1 + \cdots + 0 v_n \\ &= 0. \end{aligned}$$

So we have $U \cap W \subset \{0\}$. But U and W are subspaces of V, which means $0 \in U$ and $0 \in W$, and so we get $0 \in U \cap W$, or $\{0\} \subset U \cap W$. Therefore, we obtain the set equality $U \cap W = \{0\}$. Finally, by 1.45 of Axler, we can write $V = U \oplus W$, as desired. □

3.E.18. Suppose $T \in \mathcal{L}(V, W)$ and U is a subspace of V. Let $\pi : V \to V/U$ be the quotient map. Prove that there exists $S \in \mathcal{L}(V/U, W)$ such that $T = S \circ \pi$ if and only if $U \subset \operatorname{null} T$.

Proof. Forward direction: If there exists $S \in \mathcal{L}(V/U, W)$ such that $T = S \circ \pi$, then $U \subset \operatorname{null} T$. Suppose there exists $S \in \mathcal{L}(V/U), W)$ such that $T = S \circ \pi$. Let $u \in U$ be arbitrary. We have $v - 0 = v \in U$, and so, by 3.85—(a) implies (b)—of Axler, we have $v + U = 0 + U$. So, using 3.88 of Axler, for all $u \in U$, we have

$$\begin{aligned} Tu &= (S \circ \pi)u \\ &= S(\pi(u)) \\ &= S(u + U) \\ &= S(0 + U) \\ &= 0, \end{aligned}$$

where we also used 3.11 of Axler in the last equality above. So we have $u \in \operatorname{null} T$, and so we conclude $U \subset \operatorname{null} T$.

Backward direction: If $U \subset \operatorname{null} T$, then there exists $S \in \mathcal{L}(V/U, W)$ such that $T = S \circ \pi$. Suppose that we have $U \subset \operatorname{null} T$. Let $v \in V$ be arbitrary, and define $S : V/U \to W$ by

$$S(v + U) = Tv.$$

Consider another vector $\hat{v} \in V$ that satisfies $v + U = \hat{v} + U$. Since we assumed $U \subset \operatorname{null} T$, we have $v - \hat{v} \in \operatorname{null} T$, which means we have $T(v - \hat{v}) = 0$. So we get

$$\begin{aligned} S(v + U) &= Tv \\ &= T((v - \hat{v}) + \hat{v}) \\ &= T(v - \hat{v}) + T\hat{v} \\ &= 0 + T\hat{v} \\ &= T\hat{v} \\ &= S(\hat{v} + U), \end{aligned}$$

which means S indeed defines a function. Next, we need to show that S is linear, given already that T is linear. For all $\lambda \in \mathbb{F}$ and for all $v, w \in V$, we have

$$\begin{aligned} S((v + U) + (w + U)) &= S((v + w) + U) \\ &= T(v + w) \\ &= Tv + Tw \\ &= S(v + U) + S(w + U), \end{aligned}$$

satisfying additivity, and

$$\begin{aligned} S(\lambda(v + U)) &= S(\lambda v + U) \\ &= T(\lambda v) \\ &= \lambda Tv \\ &= \lambda S(v + U), \end{aligned}$$

satisfying homogeneity. So S is linear. Finally, for all $v \in V$, we have

$$\begin{aligned} (S \circ \pi)v &= S(\pi(v)) \\ &= S(v + V) \\ &= Tv, \end{aligned}$$

from which we conclude $T = S \circ \pi$. □

3.E.20. Suppose U is a subspace of V. Define $\Gamma : \mathcal{L}(V/U, W) \to \mathcal{L}(V, W)$ by

$$\Gamma(S) = S \circ \pi.$$

(a) Show that Γ is a linear map.

Proof. For all $\lambda \in \mathbb{F}$ and for all $S, T \in \mathcal{L}(V/U, W)$, we have

$$\begin{aligned} \Gamma(S + T) &= (S + T) \circ \pi \\ &= S \circ \pi + T \circ \pi \\ &= \Gamma(S) + \Gamma(T), \end{aligned}$$

satisfying additivity, and

$$\begin{aligned} \Gamma(\lambda S) &= (\lambda S) \circ \pi \\ &= \lambda S \circ \pi \\ &= \lambda \Gamma(S), \end{aligned}$$

satisfying homogeneity. So Γ is linear. □

(b) Show that Γ is injective.

Proof. Suppose we have $S \in \operatorname{null}\Gamma$, which means $\Gamma(S) = 0$. Then we have $S \circ \pi = \Gamma(S) = 0$, and so for all $v \in V$ we have $(S \circ \pi)v = 0$. Therefore,

$$\begin{aligned} 0 &= (S \circ \pi)v \\ &= S(\pi(v)) \\ &= S(v + U). \end{aligned}$$

Since $v \in V$ is arbitrary, we must have $S = 0$, and so $\operatorname{null}\Gamma \subset \{0\}$. But 3.14 of Axler says that $\operatorname{null}\Gamma$ is a subspace in V, which means in particular that $\operatorname{null}\Gamma$ contains the additive identity, or $\{0\} \subset \operatorname{null}\Gamma$. Therefore, we have the set equality $\operatorname{null}\Gamma = \{0\}$. Finally, by 3.16 of Axler, Γ is injective. □

(c) Show that $\operatorname{range}\Gamma = \{T \in \mathcal{L}(V, W) : Tu = 0 \text{ for all } u \in U\}$.

Proof. By Exercise 3.E.18 of Axler (or Question 2 of this examination), there exists $S \in \mathcal{L}(V/U, W)$ satisfying $T = S \circ \pi$ if and only if we have $U \subset \operatorname{null} T$. Therefore, we have

$$\begin{aligned} \operatorname{range}\Gamma &= \{\Gamma(S) \in \mathcal{L}(V, W) : S \in \mathcal{L}(V/U, W)\} \\ &= \{S \circ \pi \in \mathcal{L}(V, W) : S \in \mathcal{L}(V/U, W)\} \\ &= \{T \in \mathcal{L}(V, W) : T = S \circ \pi, S \in \mathcal{L}(V/U, W)\} \\ &= \{T \in \mathcal{L}(V, W) : U \subset \operatorname{null} T\} \\ &= \{T \in \mathcal{L}(V, W) : Tu = 0 \text{ for all } u \in U\}, \end{aligned}$$

as desired. □

3.F.3. Suppose V is finite-dimensional and $v \in V$ with $v \neq 0$. Prove that there exists $\varphi \in V'$ such that $\varphi(v) = 1$.

Proof. Since $v \in V$ is nonzero, it follows that the list v (yes, the list with one element only) is linearly independent. Furthermore, by 2.33 of Axler, we can extend this linearly independent list to a basis $v, u_1, \dots, u_n$ of V. By 3.96 of Axler, we also have a corresponding dual basis $\varphi, \varphi_1, \dots, \varphi_n$ of elements in V', with $\varphi(v) = 1$ in particular. □

3.F.4. Suppose V is finite-dimensional and U is a subspace of V such that $U \neq V$. Prove that there exists $\varphi \in V'$ such that $\varphi(u) = 0$ for every $u \in U$ but $\varphi \neq 0$.

Proof. Since V is finite-dimensional and U is a subspace of V, it follows by 2.26 of Axler that U is also finite-dimensional. By 2.32 of Axler, there exists a basis $u_1, \dots, u_m$ of U. By 2.33 of Axler, we can extend it to a basis $u_1, \dots, u_m, v_1, \dots, v_n$ of V. Because we also assumed $U \neq V$, the extension cannot be trivial; that is, we must be able to extend $u_1, \dots, u_m$ by $v_1, \dots, v_j$ for some $j \in \{1, \dots, n\}$. So there exist at least one vector, which we can call it $v_1 \in V$ without any loss of generality. This motivates us to define $\varphi : V \to \mathbb{F}$ by

$$\varphi(a_1u_1 + \cdots + a_mu_m + b_1v_1 + \cdots + b_nv_n) = b_1$$

for some $a_1, \dots, a_m, b_1, \dots, b_n \in \mathbb{F}$. According to the proof of 3.5 of Axler, this map indeed defines a function. We will prove that φ is linear.

- Additivity: If we have $v, w \in V$, then, since $u_1, \dots, u_m, v_1, \dots, v_n$ is a basis of V, we can write uniquely

$$v = a_1u_1 + \cdots + a_mu_m + b_1v_1 + \cdots + b_nv_n$$

and

$$w = c_1u_1 + \cdots + c_mu_m + d_1v_1 + \cdots + d_nv_n$$

for some $a_1, \dots, a_m, b_1, \dots, b_n, c_1, \dots, c_m, d_1, \dots, d_n \in \mathbb{F}$. So we have

$$\begin{aligned} \varphi(v + w) &= \varphi((a_1u_1 + \cdots + a_mu_m + b_1v_1 + \cdots + b_nv_n) + (c_1u_1 + \cdots + c_mu_m + d_1v_1 + \cdots + d_nv_n)) \\ &= \varphi((a_1 + c_1)u_1 \cdots + (a_m + c_m)u_m + (b_1 + d_1)v_1 + \cdots + (b_n + d_n)v_n) \\ &= b_1 + d_1 \\ &= \varphi(a_1u_1 + \cdots + a_mu_m + b_1v_1 + \cdots + b_nv_n) + \varphi(c_1u_1 + \cdots + c_mu_m + d_1v_1 + \cdots + d_nv_n) \\ &= \varphi(v) + \varphi(w). \end{aligned}$$

- Homogeneity: Suppose we have $\lambda \in \mathbb{F}$. If we have $v, w \in V$, then, since $u_1, \dots, u_m, v_1, \dots, v_n$ is a basis of V, we can write uniquely

$$v = a_1u_1 + \cdots + a_mu_m + b_1v_1 + \cdots + b_nv_n$$

for some $a_1, \dots, a_m, b_1, \dots, b_n \in \mathbb{F}$. So we have

$$\begin{aligned} \varphi(\lambda v) &= \varphi(\lambda(a_1u_1 + \cdots + a_mu_m + b_1v_1 + \cdots + b_nv_n)) \\ &= \varphi((\lambda a_1)u_1 + \cdots + (\lambda a_m)u_m + (\lambda b_1)v_1 + \cdots + (\lambda b_n)v_n) \\ &= \lambda b_1 \\ &= \lambda\varphi(a_1u_1 + \cdots + a_mu_m + b_1v_1 + \cdots + b_nv_n) \\ &= \lambda\varphi(v). \end{aligned}$$

Since additivity and homogeneity of φ are satisfied, φ is linear; in other words, we have $\varphi \in \mathcal{L}(V, \mathbb{F}) = V'$. Now, if $u \in U$, then, since $u_1, \ldots, u_m$ is a basis of U, we can write uniquely in the form

$$u = a_1 u_1 + \cdots + a_m u_m$$

for some $a_1, \ldots, a_m \in \mathbb{F}$. Therefore, for all $u \in U$, we have

$$\begin{aligned} \varphi(u) &= \varphi(a_1 u_1 + \cdots + a_m u_m) \\ &= \varphi(a_1 u_1 + \cdots + a_m u_m + 0 v_1 + \cdots + 0 v_n) \\ &= 0. \end{aligned}$$

However, if we consider the vector $v_1 \in V$, then we have $v_1 \notin U$, and more importantly, we have $b_1 \neq 0$. So we have

$$\begin{aligned} \varphi(v_1) &= \varphi(0 u_1 + \cdots + 0 u_m + 1 v_1 + 0 v_2 + \cdots + 0 v_n) \\ &= 1 \\ &\neq 0. \end{aligned}$$

In other words, we found an element $v_1 \in V$ for which φ is nonzero, and so can conclude $\varphi \neq 0$. □

3.F.6. Suppose V is finite-dimensional and $v_1, \ldots, v_m \in V$. Define a linear map $\Gamma : V' \to \mathbb{F}^m$ by

$$\Gamma(\varphi) = (\varphi(v_1), \ldots, \varphi(v_m)).$$

(a) Prove that $v_1, \ldots, v_m$ spans V if and only if Γ is injective.

Proof. Forward direction: If $v_1, \ldots, v_m$ spans V, then Γ is injective. Since $v_1, \ldots, v_m$ spans V, we can write every $v \in V$ uniquely as

$$v = a_1 v_1 + \cdots + a_m v_m$$

for some $a_1, \ldots, a_m \in \mathbb{F}$. Now, suppose we have $\varphi \in \operatorname{null} \Gamma$. Then we have $\Gamma(\varphi) = (0, \ldots, 0)$, and so we have

$$(0, \ldots, 0) = (\varphi(v_1), \ldots, \varphi(v_m)),$$

from which we can equate the coordinates of both sides to write

$$\varphi(v_1) = 0, \ldots, \varphi(v_m) = 0.$$

As we assumed throughout Section 3.F of Axler that $\varphi \in V' = \mathcal{L}(V, \mathbb{F})$, we can use its additivity and homogeneity to write

$$\begin{aligned} \varphi(v) &= \varphi(a_1 v_1 + \cdots + a_m v_m) \\ &= \varphi(a_1 v_1) + \cdots + \varphi(a_m v_m) \\ &= a_1 \varphi(v_1) + \cdots + a_m \varphi(v_m) \\ &= a_1 \cdot 0 + \cdots + a_m \cdot 0 \\ &= 0. \end{aligned}$$

Since $v \in V$ is arbitrary, we conclude that φ must be the zero map; that is, we conclude $\varphi = 0$. Therefore, we have have $\operatorname{null} \Gamma \subset \{0\}$. But 3.14 of Axler says that $\operatorname{null} \Gamma$ is a subspace in V', which means in particular that $\operatorname{null} \Gamma$ contains the additive identity, or $\{0\} \subset \operatorname{null} \Gamma$. Therefore, we have the set equality $\operatorname{null} \Gamma = \{0\}$. Finally, by 3.16 of Axler, Γ is injective.

Backward direction: If Γ is injective, then $v_1, \ldots, v_m$ spans V. Define $U = \operatorname{span}(v_1, \ldots, v_n)$, which is a subspace of V by 2.7 of Axler. Suppose by contradiction that $v_1, \ldots, v_m$ does not span V. Then we have $U \neq V$. By Exercise 3.F.4 of Axler, there exists $\varphi \in V'$ such that $\varphi(u) = 0$ for all $u \in U$ but $\varphi \neq 0$. Therefore, as Γ is linear, we have

$$\begin{aligned} \Gamma(\varphi) &= (\varphi(v_1), \ldots, \varphi(v_m)) \\ &= (0, \ldots, 0) \\ &= \Gamma(0). \end{aligned}$$

In other words, we found the zero map $0 \in V'$ and a nonzero functional $\varphi \in V'$ such that $\Gamma(0) = 0$ and $\Gamma(\varphi) = 0$. This signifies that Γ is not injective, which contradicts our assumption that Γ is injective. Therefore, $v_1, \ldots, v_m$ spans V. □

(b) Prove that $v_1, \ldots, v_m$ is linearly independent if and only if Γ is surjective.

Proof. Forward direction: If $v_1, \ldots, v_m$ is linearly independent, then Γ is surjective. Since $v_1, \ldots, v_m$ is linearly independent, by 2.33 of Axler, the list extends to a basis $v_1, \ldots, v_m, u_1, \ldots, u_n$ of V. Define $\varphi : V \to \mathbb{F}$ by

$$\varphi(a_1 u_1 + \cdots + a_m u_m + b_1 v_1 + \cdots + b_n v_n) = a_j x_j$$

for some $a_1, \dots, a_m, b_1, \dots, b_n \in \mathbb{F}$, for all $x_j \in \mathbb{F}$ and for any $j = 1, \dots, m$. Then we can consider an arbitrary vector $(x_1, \dots, x_m) \in \mathbb{F}^m$. According to the proof of 3.5 of Axler, this map indeed defines a function. Also, for all $j = 1, \dots, m$, we have

$$\begin{aligned}\varphi(v_j) &= \varphi(0u_1 + \cdots + 0u_{j-1} + 1u_j + 0u_{j+1} + \cdots + 0u_m + 0v_1 + \cdots + 0v_n) \\ &= 1x_j \\ &= x_j.\end{aligned}$$

Therefore, we have

$$\begin{aligned}\Gamma(\varphi) &= (\varphi(v_1), \dots, \varphi(v_m)) \\ &= (x_1, \dots, x_m),\end{aligned}$$

and so we get $(x_1, \dots, x_m) \in \operatorname{range}\Gamma$. So we have $\mathbb{F}^m \subset \operatorname{range}\Gamma$. But 3.19 of Axler states that $\operatorname{range}\Gamma$ is a subspace of $\mathbb{F}^m$. Therefore, we conclude the set equality $\operatorname{range}\Gamma = \mathbb{F}^m$, which means Γ is surjective.

Backward direction: If Γ is surjective, then $v_1, \dots, v_m$ is linearly independent. Suppose by contradiction that $v_1, \dots, v_m$ is linearly dependent. By the Linear Dependence Lemma (2.21 of Axler), there exists $j \in \{1, \dots, m\}$ such that we have $v_j \in \operatorname{span}(v_1, \dots, v_{j-1})$. In other words, we can write

$$v_j = -\frac{a_1}{a_j}v_1 - \cdots - \frac{a_{j-1}}{a_j}v_{j-1}$$

for some $a_1, \dots, a_j \in \mathbb{F}$ and for some $j \in \{1, \dots, m\}$. So we have

$$\begin{aligned}\varphi(v_j) &= \varphi\left(-\frac{a_1}{a_j}v_1 - \cdots - \frac{a_{j-1}}{a_j}v_{j-1}\right) \\ &= \varphi\left(-\frac{a_1}{a_j}v_1 - \cdots - \frac{a_{j-1}}{a_j}v_{j-1} + 0v_j + \cdots + 0v_n\right) \\ &= 0x_j \\ &= 0.\end{aligned}$$

In other words, we have

$$\begin{aligned}\Gamma(\varphi) &= (\varphi(v_1), \dots, \varphi(v_m)) \\ &= (\varphi(v_1), \dots, \varphi(v_{j-1}), \varphi(v_j), \varphi(v_{j+1}), \dots, \varphi(v_m)) \\ &= (\varphi(v_1), \dots, \varphi(v_{j-1}), 0, \varphi(v_{j+1}), \dots, \varphi(v_m)),\end{aligned}$$

where the 0 appearing in the last expression of $\Gamma(\varphi)$ is placed at the j^{th} coordinate of the vector in $\mathbb{F}^m$. This implies, for example, that we have $(0, \dots, 0, 1, 0, \dots, 0) \notin \operatorname{range}\Gamma$, where the 1 is placed at the j^{th} coordinate of the vector in $\mathbb{F}^m$, because we cannot, for example, set $\varphi(v_j) = 1$ when we just established $\varphi(v_j) = 0$ above. As soon as we discover at least one element in $\mathbb{F}^m$—such as $(0, \dots, 0, 1, 0, \dots, 0)$—that does not belong to $\operatorname{range}\Gamma$, we conclude $\operatorname{range}\Gamma \neq \mathbb{F}^m$, and so Γ is not surjective. But this contradicts our assumption that Γ is surjective. So we conclude that $v_1, \dots, v_m$ is linearly independent. □

3.F.7. Suppose m is a positive integer. Show that the dual basis of the basis $1, x, \dots, x^m$ of $\mathcal{P}_m(\mathbb{R})$ is $\varphi_0, \varphi_1, \dots, \varphi_m$, where $\varphi_j(p) = \frac{p^{(j)}(0)}{j!}$. Here, $p^{(j)}$ denotes the j^{th} derivative of p, with the understanding that the 0^{th} derivative of p is p.

Proof. Let $j, k = 0, 1, \dots, m$. To show that $\varphi_1, \dots, \varphi_n$ is a dual basis of V', we need to satisfy Definition 2.96 of Axler, which states that φ_j satisfies

$$\varphi_j(v_k) = \begin{cases} 1 & \text{if } k = j, \\ 0 & \text{if } k \neq j. \end{cases}$$

To prove this, we must consider the three cases $k < j, k = j, k > j$. If $k < j$, then

$$\begin{aligned}\varphi_j(v_k) &= \frac{\frac{d^j}{dx^j}x^k|_{x=0}}{j!} \\ &= \frac{\frac{d^{j-k}}{dx^{j-k}}\frac{d^k}{dx^k}x^k|_{x=0}}{j!} \\ &= \frac{\frac{d^{j-k}}{dx^{j-k}}k!|_{x=0}}{j!} \\ &= 0.\end{aligned}$$

If $k = j$, then

$$\begin{aligned}
\varphi_j(v_k) &= \varphi_j(v_j) \\
&= \frac{\frac{d^j}{dx^j} x^j|_{x=0}}{j!} \\
&= \frac{j! x^{j-j}|_{x=0}}{j!} \\
&= \frac{j! \cdot 1|_{x=0}}{j!} \\
&= 1.
\end{aligned}$$

If $k > j$, then

$$\begin{aligned}
\varphi_j(v_k) &= \frac{\frac{d^j}{dx^j} x^k|_{x=0}}{j!} \\
&= \frac{k(k-1)\cdots(k-j)x^{k-j}|_{x=0}}{j!} \\
&= \frac{k(k-1)\cdots(k-j)(0)^{k-j}}{j!} \\
&= 0.
\end{aligned}$$

So we satisfied Definition 3.96 of Axler for $\varphi_j(v_k)$. We conclude that $\varphi_1, \dots, \varphi_n$ is the dual basis of the basis in part (a). □

3.F.8. Suppose m is a positive integer.

(a) Show that $1, x-5, \dots, (x-5)^m$ is a basis of $\mathcal{P}_m(\mathbb{R})$.

Proof. Suppose $a_0, a_1, \dots, a_m \in \mathbb{F}$ satisfy

$$a_0 + a_1(x-5) + \cdots + a_m(x-5)^m = 0.$$

We will follow Example 2.41 of Axler, which does not require expanding out the polynomials. For all $j = 1, \dots, m$, we see that the left-hand side of the above equation has an $a_j(x-5)^j$ term but the right-hand side does not, which implies $a_j = 0$. In other words, we have $a_1 = 0, \dots, a_m = 0$. The above equation $a_0 + a_1(x-5) + \cdots + a_m(x-5)^m = 0$ with $a_1 = 0, \dots, a_m = 0$ implies $a_0 = 0$. So we have

$$a_0 = 0, a_1 = 0, \dots, a_m = 0,$$

and so the list $1, x-5, \dots, (x-5)^m$ is linearly independent. Finally, since the length of the list $1, x-5, \dots, (x-5)^m$ has length $m+1$ and we have $\dim \mathcal{P}^m(\mathbb{R}) = m+1$, by 2.39 of Axler, the list $1, x-5, \dots, (x-5)^m$ is a basis of $\mathcal{P}^m(\mathbb{R})$. □

(b) What is the dual basis of the basis in part (a)?

Proof. Define $\varphi : \mathcal{P}^m(\mathbb{R}) \to \mathbb{R}$ by

$$\varphi_j(p) = \frac{p^{(j)}(5)}{j!},$$

where $p^{(j)}$ is the j^{th}-order derivative of $p \in \mathcal{P}^m(\mathbb{R})$. According to the definition of the dual basis (3.96 of Axler), for all $j, k = 1, \dots, m$, we must check that φ_j satisfies

$$\varphi_j(v_k) = \begin{cases} 1 & \text{if } k = j, \\ 0 & \text{if } k \neq j. \end{cases}$$

where $v_1 = 1, v_2 = x-5, \dots, v_m = (x-5)^m$. To prove this, we must consider the three cases $k < j, k = j, k > j$. If $k < j$, then

$$\begin{aligned}
\varphi_j(v_k) &= \frac{\frac{d^j}{dx^j}(x-5)^k|_{x=5}}{j!} \\
&= \frac{\frac{d^{j-k}}{dx^{j-k}} \frac{d^k}{dx^k}(x-5)^k|_{x=5}}{j!} \\
&= \frac{\frac{d^{j-k}}{dx^{j-k}} k!|_{x=5}}{j!} \\
&= 0.
\end{aligned}$$

If $k = j$, then

$$\begin{aligned}\varphi_j(v_k) &= \varphi_j(v_j)\\ &= \frac{\frac{d^j}{dx^j}(x-5)^j|_{x=5}}{j!}\\ &= \frac{j!(x-5)^{j-j}|_{x=5}}{j!}\\ &= \frac{j!\cdot 1|_{x=5}}{j!}\\ &= 1.\end{aligned}$$

If $k > j$, then

$$\begin{aligned}\varphi_j(v_k) &= \frac{\frac{d^j}{dx^j}(x-5)^k|_{x=5}}{j!}\\ &= \frac{k(k-1)\cdots(k-j)(x-5)^{k-j}|_{x=5}}{j!}\\ &= \frac{k(k-1)\cdots(k-j)(5-5)^{k-j}}{j!}\\ &= 0.\end{aligned}$$

We conclude that $\varphi_1, \ldots, \varphi_n$ is the dual basis of the basis in part (a). □

3.F.15. Suppose W is finite-dimensional and $T \in \mathcal{L}(V, W)$. Prove that $T' = 0$ if and only if $T = 0$.

Proof. Forward direction: If $T' = 0$, then $T = 0$. Suppose we have $T' = 0$. Then, for all $\varphi \in V'$, we have $T(\varphi) = 0$. Suppose by contradiction that we have $T \neq 0$. Then there exists $v \in V$ such that $Tv \neq 0$. By Exercise 3.F.3 of Axler (or part (a) of this question), there exists $\varphi \in V'$ such that $\varphi(Tv) = 1$. In fact, by 3.99 of Axler, we have

$$\begin{aligned}T'(\varphi(v)) &= \varphi \circ T(v)\\ &= \varphi(Tv)\\ &= 1.\end{aligned}$$

But this contradicts our assumption that, for all $\varphi \in V'$, we have $T'(\varphi) = 0$. So we conclude $T = 0$.

Backward direction: If $T = 0$, then $T' = 0$. Suppose we have $T = 0$. Thenk for all $\varphi \in W'$, the dual map of T is $T'(\varphi) = \varphi \circ T$. Since $T = 0$, for all $\varphi \in W'$, we have

$$\begin{aligned}T'(\varphi) &= \varphi \circ T\\ &= \varphi(0)\\ &= 0.\end{aligned}$$

So we conclude $T' = 0$. □

3.F.32. Suppose $T \in \mathcal{L}(V)$, and $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ are bases of V. Prove that the following are equivalent:

(a) T is invertible.
(b) The columns of $\mathcal{M}(T)$ are linearly independent in $\mathbb{F}^{n,1}$.
(c) The columns of $\mathcal{M}(T)$ span $\mathbb{F}^{n,1}$.
(d) The rows of $\mathcal{M}(T)$ are linearly independent in $\mathbb{F}^{1,n}$.
(e) The rows of $\mathcal{M}(T)$ span $\mathbb{F}^{1,n}$.

Here $\mathcal{M}(T)$ means $\mathcal{M}(T, (u_1, \ldots, u_n), (v_1, \ldots, v_n))$.

Proof. According to 3.69 of Axler, statement (a) holds if and only if T is injective and if and only if T is surjective. Therefore, to establish that all the statements are equivalent, it remains to prove the following:

- T is injective if and only if statement (b) holds,
- Statement (b) is equivalent to statement (d),
- T is surjective if and only if statement (c) holds,
- Statement (c) is equivalent to statement (e).

The first two statements in the above list will establish that statement (a) is equivalent to statements (b) and (d). The last two statements in the above list will establish that statement (a) is equivalent to statements (c) and (e). These results signify that proving the above list means proving that statements (a), (b), (c), (d), (e) are equivalent to each other, as desired. To facilitate our proofs of the statements in the above list, we represent the matrix $\mathcal{M}(T) \in \mathbb{F}^{n,n}$ with respect to the bases $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ of V as

$$\mathcal{M}(T) = \begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & \ddots & \vdots \\ A_{n,1} & \cdots & A_{n,n} \end{pmatrix}$$

with its entries $A_{j,k} \in \mathbb{F}$, for any $j, k = 1, \ldots, n$. Since $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ are bases of V, it follows that, according to 3.60 of Axler, $\mathcal{M}$ is an isomorphism between $\mathcal{L}(V, V)$ and $\mathbb{F}^{n,n}$.

First, we will prove that T is injective if and only if statement (b) holds. By 3.16 of Axler, T is injective if and only if we have $\operatorname{null} T = \{0\}$. And we have $\operatorname{null} T = \{0\}$ if and only if the equation $Tu = 0$ has only the trivial solution $u = 0$. This is equivalent to saying that

$$\mathcal{M}(T)\mathcal{M}(u) = \mathcal{M}(Tu) = \mathcal{M}(0),$$

or equivalently

$$\begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & \ddots & \vdots \\ A_{n,1} & \cdots & A_{n,n} \end{pmatrix} \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix},$$

has only the trivial solution $c_1 = \cdots = c_n = 0$. This is equivalent to saying that the equation

$$\begin{aligned} c_1\mathcal{M}(T)_{\cdot,1} + \cdots + c_n\mathcal{M}(T)_{\cdot,n} &= c_1 A_{\cdot,1} + \cdots + c_n A_{\cdot,n} \\ &= c_1 \begin{pmatrix} A_{1,1} \\ \vdots \\ A_{n,1} \end{pmatrix} + \cdots + c_n \begin{pmatrix} A_{n,1} \\ \vdots \\ A_{n,n} \end{pmatrix} \\ &= \begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & \ddots & \vdots \\ A_{n,1} & \cdots & A_{n,n} \end{pmatrix} \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \end{aligned}$$

has only the trivial solution $c_1 = 0, \ldots, c_n = 0$. This is equivalent to saying that $\mathcal{M}(T)_{\cdot,1}, \ldots, \mathcal{M}(T)_{\cdot,n}$—the list of columns of $\mathcal{M}(T)$—is linearly independent in $\mathbb{F}^{n,1}$, which is statement (b).

Next, we will prove that statement (b) is equivalent to statement (d). According to the previous paragraph, statement (b) holds if and only if the matrix equation

$$\mathcal{M}(T)\mathcal{M}(u) = \mathcal{M}(0)$$

has only the trivial solution $u = 0$. Taking transposes of both sides of the matrix equation, we find that our previous statement is equivalent to saying

$$\begin{aligned} (\mathcal{M}(u))^t(\mathcal{M}(T))^t &= (\mathcal{M}(T)\mathcal{M}(u))^t \\ &= (\mathcal{M}(0))^t, \end{aligned}$$

or equivalently

$$\begin{pmatrix} c_1 & \cdots & c_n \end{pmatrix} \begin{pmatrix} A_{1,1} & \cdots & A_{n,1} \\ \vdots & \ddots & \vdots \\ A_{1,n} & \cdots & A_{n,n} \end{pmatrix} = \begin{pmatrix} 0 & \cdots & 0 \end{pmatrix},$$

has only the trivial solution $u = 0$. This is equivalent to saying that the equation

$$\begin{aligned} c_1\mathcal{M}(T)_{1,\cdot} + \cdots + c_n\mathcal{M}(T)_{n,\cdot} &= c_1 A_{1,\cdot} + \cdots + c_n A_{n,\cdot} \\ &= c_1 \begin{pmatrix} A_{1,1} & \cdots & A_{n,1} \end{pmatrix} + \cdots + c_n \begin{pmatrix} A_{n,1} & \cdots & A_{n,n} \end{pmatrix} \\ &= \begin{pmatrix} c_1 & \cdots & c_n \end{pmatrix} \begin{pmatrix} A_{1,1} & \cdots & A_{n,1} \\ \vdots & \ddots & \vdots \\ A_{1,n} & \cdots & A_{n,n} \end{pmatrix} \\ &= \begin{pmatrix} 0 & \cdots & 0 \end{pmatrix} \end{aligned}$$

has only the trivial solution $c_1 = 0, \ldots, c_n = 0$. This is equivalent to saying that $\mathcal{M}(T)_{1,\cdot}, \ldots, \mathcal{M}(T)_{n,\cdot}$—the list of rows of $\mathcal{M}(T)$—is linearly independent in $\mathbb{F}^{n,1}$, which is statement (d).

Next, we will prove that T is surjective if and only if statement (c) holds. By definition, T is surjective if and only if we have $\operatorname{range} T = V$. According to the proof of the Fundamental Theorem of Linear Maps (3.22 of Axler), the list $Tu_1, \ldots, Tu_n$ is a basis of V, which means in particular that the list spans V, and so we have $\operatorname{span}(Tu_1, \ldots, Tu_n) = \operatorname{range} T$. Therefore, T is surjective if and only if we have

$$\begin{aligned} V &= \operatorname{range} T \\ &= \operatorname{span}(Tu_1, \ldots, Tu_n). \end{aligned}$$

This is equivalent to saying that, for all $v \in V$, we have

$$\begin{aligned} v &= Tu \\ &= c_1 Tu_1 + \cdots + c_n Tu_n \end{aligned}$$

for some $c_1, \ldots, c_n \in \mathbb{F}$. Using their matrix representations, the vector v written as a linear combination of the basis vectors $Tu_1, \ldots, Tu_n$ of $\operatorname{range} T$ is equivalent to saying

$$\begin{aligned} \mathcal{M}(v) &= \mathcal{M}(c_1 Tu_1 + \cdots + c_n Tu_n) \\ &= \mathcal{M}(c_1 Tu_1) + \cdots + \mathcal{M}(c_n Tu_n) \\ &= c_1 \mathcal{M}(Tu_1) + \cdots + c_n \mathcal{M}(Tu_n) \\ &= c_1 \mathcal{M}(T)_{\cdot,1} + \cdots + c_n \mathcal{M}(T)_{\cdot,n} \end{aligned}$$

for some $c_1, \ldots, c_n \in \mathbb{F}$. This is equivalent to saying that $\mathcal{M}(T)_{\cdot,1}, \ldots, \mathcal{M}(T)_{\cdot,n}$—the list of columns of $\mathcal{M}(T)$—spans $\mathbb{F}^{n,1}$, which is statement (c).

Finally, we will prove that statement (c) is equivalent to statement (e). According to the previous paragraph, statement (c) holds if and only if we have the matrix equation

$$\mathcal{M}(v) = c_1 \mathcal{M}(T)_{\cdot,1} + \cdots + c_n \mathcal{M}(T)_{\cdot,n}$$

for some $c_1, \ldots, c_n \in \mathbb{F}$. Taking transposes of both sides of the matrix equation, we find that our previous statement is equivalent to saying

$$\begin{aligned} (\mathcal{M}(v))^t &= (c_1 \mathcal{M}(T)_{\cdot,1} + \cdots + c_n \mathcal{M}(T)_{\cdot,n})^t \\ &= (c_1 A_{\cdot,1} + \cdots + c_n A_{\cdot,n})^t \\ &= \left(\begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & \ddots & \vdots \\ A_{n,1} & \cdots & A_{n,n} \end{pmatrix} \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} \right)^t \\ &= \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}^t \begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & \ddots & \vdots \\ A_{n,1} & \cdots & A_{n,n} \end{pmatrix}^t \\ &= \begin{pmatrix} c_1 & \cdots & c_n \end{pmatrix} \begin{pmatrix} A_{1,1} & \cdots & A_{n,1} \\ \vdots & \ddots & \vdots \\ A_{1,n} & \cdots & A_{n,n} \end{pmatrix} \\ &= \begin{pmatrix} c_1 A_{1,1} + \cdots + c_n A_{n,1} & \cdots & c_1 A_{1,n} + \cdots + c_n A_{n,n} \end{pmatrix} \\ &= \begin{pmatrix} c_1 A_{1,1} & \cdots & c_1 A_{1,n} \end{pmatrix} + \cdots + \begin{pmatrix} c_n A_{n,1} & \cdots & c_n A_{n,n} \end{pmatrix} \\ &= c_1 \begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \end{pmatrix} + \cdots + c_n \begin{pmatrix} A_{n,1} & \cdots & A_{n,n} \end{pmatrix} \\ &= c_1 A_{1,\cdot} + \cdots + c_n A_{n,\cdot} \\ &= c_1 \mathcal{M}(T)_{1,\cdot} + \cdots + c_n \mathcal{M}(T)_{n,\cdot} \end{aligned}$$

for some $c_1, \ldots, c_n \in \mathbb{F}$. This is equivalent to saying that $\mathcal{M}(T)_{1,\cdot}, \ldots, \mathcal{M}(T)_{n,\cdot}$—the list of rows of $\mathcal{M}(T)$—spans $\mathbb{F}^{1,n}$, which is statement (e). □