Math 131

6/24 Complex Number
- ordered pair (a,b) $a,b \in \mathbb{R}$ $\alpha = a+ib$

- The set of ALL complex numbers:
$\mathbb{C} = \{a+ib \mid a,b \in \mathbb{R}\}$

- Addition & Multiplication on $\mathbb{C}$:
$(a+ib)+(c+di) = (a+c)+i(b+d)$
$(a+ib)(c+di) = (ac-bd)+i(ad+bc)$

Example)
evaluate: $(2+3i)(4+5i)$
$8+12i+10i-15$
$\boxed{-7+22i}$

Properties of $\mathbb{C}$
$\alpha, \beta \in \mathbb{C}$

① ~~commuta~~ Commutativity
$\alpha+\beta = \beta+\alpha$, $\alpha\beta = \beta\alpha$ $\forall \alpha,\beta \in \mathbb{C}$ for ALL

$\alpha+\beta=\beta+\alpha$ Proof) $\alpha = a+ib$
$\beta = c+id$
$\alpha+\beta = (a+ib)+(c+id) = (a+c)+i(b+d)$
$= (c+a)+i(d+b) =$
$(c+id)+(a+ib) = \beta+\alpha$

$\alpha\beta=\beta\alpha$ Proof) $\alpha = a+ib$
$\beta = c+id$
$\alpha\beta = (a+ib)(c+id) = ac+cib+aid-bd$
$= (ac-bd)+i(cb+ad)$
$\beta\alpha = (c+id)(a+ib)$ $= ca+aid+cib-bd$ $\|$ ✓
$(ca-bd)+i(ad+cb)$

② Associativity

$(\alpha+\beta)+\lambda = \alpha+(\beta+\lambda) \quad \forall\, \alpha, \beta, \lambda \in \mathbb{C}$

$(\alpha\beta)\lambda = \alpha(\beta\lambda) \quad \forall\, \alpha, \beta, \lambda \in \mathbb{C}$

③ identitties

$\lambda + 0 = \lambda$

$\lambda \cdot 1 = \lambda \quad \forall \lambda = \mathbb{C}$

④ additive & multiplicative Inverse.

$\forall\, \alpha = \mathbb{C},\ \exists!\ \beta \in \mathbb{C}\ ST\ \alpha+\beta=0$

for any — there exists — unique — such that

$\forall\, \alpha \in \mathbb{C}\quad \exists!\ \beta \in \mathbb{C}\ ST\ \alpha\beta = 1.$

$\alpha \neq 0$

⑤ $\lambda(\alpha+\beta) = \lambda\alpha + \lambda\beta \quad \forall\, \lambda, \alpha, \beta \in \mathbb{C}$

Definition) $\alpha, \beta \in \mathbb{C}$

1. $(-\alpha)$ denotes additive inverse of α.

   $\alpha + (-\alpha) = 0$

2. Subtraction

   $\beta - \alpha = \beta + (-\alpha)$

3. for $\alpha \neq 0$

   let $\frac{1}{\alpha}$ be multiplicative inverse of α.

   $\alpha\left(\frac{1}{\alpha}\right) = 1$

NOTE) $\alpha = a+ib$

$\frac{1}{\alpha} = \frac{a}{a^2+b^2} - \frac{b}{a^2+b^2} i$

Proof) $\alpha\left(\frac{1}{\alpha}\right) = 1 \qquad \alpha = a+bi$

$1/\alpha = \frac{a}{a^2+b^2} - \frac{b}{a^2+b^2} i$

$\alpha\left(\frac{1}{\alpha}\right) = (a+bi)\left(\frac{a}{a^2+b^2} - \frac{b}{a^2+b^2} i\right)$

$= \frac{a^2}{a^2+b^2} - \cancel{\frac{ab}{a^2+b^2} i} + \cancel{\frac{ab}{a^2+b^2} i} + \frac{b^2}{a^2+b^2} = \frac{a^2+b^2}{a^2+b^2} = 1$

$\boxed{\alpha\left(\frac{1}{\alpha}\right) = 1}$

Notation $\mathbb{F}$ stands for $\mathbb{C}$ or $\mathbb{R}$

field complex real

- Elements of $\mathbb{F}$ is $\boxed{\text{Scalar}}$

$\boxed{\text{Def}}$ $a \in \mathbb{F}$, $m \in \mathbb{N}$

$a^m = a_1 \cdot a \cdots a \cdot a \cdot a_m$

$\boxed{\text{Note}}$ $(a^m)^n = a^{mn}$ $\forall\ a, \beta \in \mathbb{F}$

$(a\beta)^n = a^n \beta^n$ $\quad m, n \in \mathbb{N}$

list = order matters

sets = order does not matter

Set ex) $\{3,4\} = \{4,3\}$

list ex) $(3,4) \neq (4,3)$

$\boxed{\text{Addition in } \mathbb{F}^n}$

$x, y \in \mathbb{F}^n$ $\quad \forall i \in \{1, \ldots, n\}$

$x = (x_1 \ldots x_n)$ $\quad x_i, y_i \in \mathbb{F}$

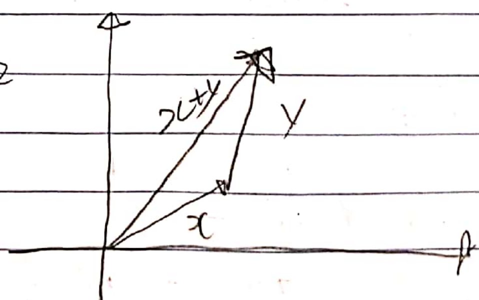
$y = (y_1 \ldots y_n)$

$x + y = (x_1 \ldots x_n) + (y_1 \ldots y_n)$

$= (x_1 + y_1 \ldots x_n + y_n)$

$\boxed{\text{Note}}$

$x, y \in \mathbb{R}^2$



$\boxed{\text{Def}}$ Scalar Multiplication:

$\lambda \in \mathbb{F}, (x_1 \ldots x_n) \in \mathbb{F}^n$

$\lambda(x_1 \ldots x_n) = (\lambda x_1 \ldots \lambda x_n)$

Ex)
$\lambda = 2-i \quad x \in \mathbb{C}^2$ ST $x = (3-2i, 5i)$

Find $\lambda x = (2-i)(3-2i, 5i)$
$= (6-3i-4i-2, 10i+5)$
$= (4-7i, 5+10i)$

Check 1.A
4, 5, 6, 9, 10, 11, 12, 13, 15, 16

6/25 1.B Vector Space $(V, +, \cdot)$

set of vectors, add, scalar multiplication

Def) Vector Space consists of a set of Vectors and a field F.

- vectors can be added to create another vector.
  $\forall \vec{u}, \vec{v} \in V, \vec{u} + \vec{v} \in V$
- Scalar can be multiplied with the a vector to a new vector as well.

★ Add & Scalar mult. must satisfy the following:

1. commutativity: $\vec{u}+\vec{v} = \vec{v}+\vec{u} \quad \forall \vec{u}, \vec{v} \in V$
2. Associativity: $(\vec{u}+\vec{v})+\vec{w} = \vec{u}+(\vec{v}+\vec{w}) \quad \forall \vec{u}, \vec{v}, \vec{w} \in V$
   $(\alpha B)\vec{v} = \alpha(B\vec{v}) \quad \forall \alpha, B \in F$
3. Add Identity $\exists \vec{0} \in V$ ST $\vec{0}+\vec{v} = \vec{v} \quad \forall \vec{v} = V$
4. dd Inverse $\forall \vec{v} \in V, \exists \vec{w} \in V$ ST $\vec{v}+\vec{w} = \vec{0}$
   note: $(\vec{w} = -\vec{v})$
5. multi. identity $1\vec{v} = \vec{v} \quad \forall \vec{v} \in V$
6. Distributive prop. $\alpha(\vec{u}+\vec{v}) = \alpha\vec{u} + \alpha\vec{v}$
   $(\alpha + B)\vec{v} = \alpha\vec{v} + B\vec{v}$
   $\forall \vec{u}, \vec{v} \in V$
   $\alpha, B \in F$