Lec. 01

Def. Complex numbers.

- A complex number is an ordered pair (a, b)
  $a, b \in \mathbb{R}$, $\quad \alpha = a + ib$
- The set of all complex numbers:
  $$\mathbb{C} = \{a+ib \mid a, b \in \mathbb{R}\}$$
- Addition and multiplications on $\mathbb{C}$:
  $$\underbrace{(a+ib)}_{\in \mathbb{C}} + \underbrace{(c+id)}_{\in \mathbb{C}} = (a+c) + i(b+d)$$
  $$(a+ib)(c+id) = (ac - bd) + i(ad + bc)$$

Remark: "i" is a solution of the equation $x^2 = -1$. Because there is no real number satisfies this equation, "i" is called an "imaginary number".

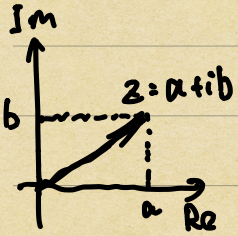
We can verify that $i^2 = -1$ is consistent with def.

Ex. evaluate $(2+3i)(4+5i)$
$$= (8 - 15) + i(10+12)$$
$$= -7 + 22i$$

$Z = a + bi \qquad a = \mathrm{Re}(Z) \qquad b = \mathrm{Im}(Z)$

Ex: $\mathrm{Re}(Z) = 2 \quad \mathrm{Im}(Z) = -1 \quad \Rightarrow Z = 2 - i$

Visualization: $z = a + ib \in \mathbb{C}$ can be seen as a coordinate of a point in a two-dim space.



$z = (a, b)$

Properties of $\mathbb{C}$; $(\alpha, \beta \in \mathbb{C})$

- Commutivity

For all

$\alpha + \beta = \beta + \alpha$, $\alpha\beta = \beta\alpha$, $\forall$ $\alpha, \beta \in \mathbb{C}$.

proof: $\alpha = a + ib$ $\beta = c + id$

$$\alpha + \beta = (a+ib) + (c+id) = (a+c) + i(b+d)$$
$$= (c+a) + i(d+b)$$
$$= (c+id) + (a+ib) = \beta + \alpha$$

$\alpha\beta = \beta\alpha$ $\forall \alpha, \beta = \mathbb{C}$

$$\alpha\beta = (a+ib)(c+id) = ac - bd + i(ad + bc)$$
$$= (c+id)(a+ib) = \beta\alpha$$

- associativity

$(\alpha + \beta) + \lambda = \alpha + (\beta + \lambda)$ $\forall$ $\alpha, \beta, \lambda \in \mathbb{C}$

$(\alpha\beta)\lambda = \alpha(\beta\lambda)$ $\forall$ $\alpha, \beta, \lambda \in \mathbb{C}$

- identities

$\lambda + 0 = \lambda$

$\lambda \cdot 1 = \lambda$ $\forall \lambda \in \mathbb{C}$.

- additive & multiplication Inverse.

$\forall \alpha \in \mathbb{C}, \ \exists ! \ \beta \in \mathbb{C}$ s.t. $\alpha + \beta = 0$

unique   such that

$\forall \underset{\alpha \neq 0}{\alpha \in \mathbb{C}}, \ \exists ! \ \beta \in \mathbb{C}$ s.t. $\alpha \cdot \beta = 1$

- $\lambda(\alpha + \beta) = \lambda\alpha + \lambda\beta, \ \forall \lambda, \alpha, \beta \in \mathbb{C}$

Def. $\alpha, \beta \in \mathbb{C}$

i) $(-\alpha)$ denotes the additive inverse of α.

$\alpha + (-\alpha) = 0$

Remark: $\alpha = a + ib, \ -\alpha = (-a) + i(-b)$

ii) subtraction:

$\beta - \alpha = \beta + (-\alpha)$

iii) for $\alpha \neq 0$

let $1/\alpha$ be the multiplicative inverse of α.

$\alpha\left(\frac{1}{\alpha}\right) = 1$

Remark: $\alpha = a + ib$, then

$\frac{1}{\alpha} = \frac{a}{a^2+b^2} - \frac{b}{a^2+b^2} i$

proof: WTS $\alpha\left(\frac{1}{\alpha}\right) = 1$

$\alpha\left(\frac{1}{\alpha}\right) = (a + ib)\left(\frac{a}{a^2+b^2} - \frac{b}{a^2+b^2} i\right)$

$= \frac{a^2}{a^2+b^2} - \frac{abi}{a^2+b^2} + \frac{abi}{a^2+b^2} - \frac{b^2 i^2}{a^2+b^2}$

$= \frac{a^2+b^2}{a^2+b^2}$

$= i$

Notation: $\underset{\substack{\uparrow \\ \text{field}}}{F}$ stands for $\mathbb{C}$ or $\mathbb{R}$

- Elements of F are called scalars.
- Def. $\alpha \in F$, $m \in \mathbb{N}$

$$\alpha^m = \underbrace{\alpha \cdot \alpha \cdot \cdots \cdot \alpha}_{m \text{ times}}$$

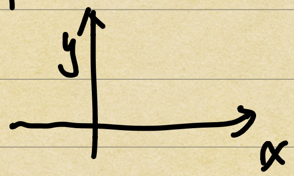
one can show that:

$(\alpha^m)^n = \alpha^{mn}$ $\qquad \forall\, \alpha, \beta \in F$

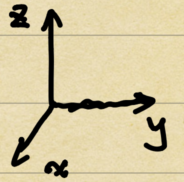
$(\alpha\beta)^n = \alpha^n \beta^n$ $\qquad m, n \in \mathbb{N}$

Ex: $\alpha = 3+4i$ $\qquad \frac{1}{\alpha} = \frac{3}{25} - \frac{4}{25} i$

Ex: $\mathbb{R}^2 = \{(x, y) \mid x, y \in \mathbb{R}\} \rightarrow$



$\mathbb{R}^3 = \{(x, y, z) \mid x, y, z \in \mathbb{R}\} \rightarrow$



Def: A list of length n: $(x_1, x_2, \cdots, x_n)$

two lists are equally iff they have same length and the same elements in the same order.

$x = (x_1, x_2, \cdots, x_n)$ $\qquad y = (y_1, y_2, \cdots, y_n)$

$x = y \Leftrightarrow x_i = y_i \quad \forall i \in \{1, \cdots, n\}$

Ex. sets both $\{3, 4\}$ and $\{4, 3\}$ are equal!

But. lists: $(3, 4) \neq (4, 3)$

$\mathbb{R}$

$\mathbb{R}^2 = \{(a,b) \mid a,b \in \mathbb{R}\}$

$\mathbb{R}^n = \{(x_1, x_2, \cdots, x_n) \mid x_i \in \mathbb{R}\}$

$\mathbb{C}^n = \{(x_1, x_2, \cdots, x_n) \mid x_i \in \mathbb{C}\}$

Def: $F^n = \{(x_1, x_2, \cdots, x_n) \mid x_i \in F,\ i \in \{1, 2, \cdots, n\}\}$

Ex: $\mathbb{C}^4 = \{(z_1, z_2, z_3, z_4) \mid z_i \in \mathbb{C} \quad i \in \{1,2,3,4\}\}$

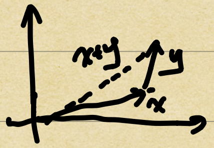
Def: Addition in F^n: $\quad x, y \in F^n$

$x = (x_1, x_2 \cdots x_n)$

$y = (y_1, y_2 \cdots y_n) \qquad x_i, y_i \in F$

$$\begin{aligned} x + y &= (x_1, \cdots, x_n) + (y_1, \cdots, y_n) \\ &= (x_1 + y_1, x_2 + y_2, \cdots, x_n + y_n) \end{aligned}$$

Geometric Intuition: $\quad x, y \in \mathbb{R}^2$



Def: scalar multiplication

$\lambda \in F$, $(x_1, x_2, \cdots, x_n) \in F^n$,

$\lambda(x_1, x_2, \cdots, x_n) = (\lambda x_1, \lambda x_2, \cdots, \lambda x_n)$

Ex: $\lambda = 2-i$, $x \in \mathbb{C}^2$ s.t. $x = (3-2i, 5i)$

Find $\lambda x = (2-i)(3-2i, 5i)$

$= (4-7i, 5+10i)$

Ex: 1.A 4, 5, 6, 9, 10, 11, 12, 13, 15, 16.