6/24/19 **1A;**

Def. Complex numbers;

- A complex number is a ordered pair (a, b) $a, b \in \mathbb{R}$, $\alpha = a + ib$
- The set of all complex numbers: $\mathbb{C} := \{a + ib \mid a, b \in \mathbb{R}\}$
- addition and multiplication on $\mathbb{C}$:

$$\underbrace{(a+ib)}_{\in \mathbb{C}} + \underbrace{(c+id)}_{\in \mathbb{C}} := (a+c) + i(b+c)$$

$$(a+ib)(c+id) := (ac-bd) + i(ad+bc)$$

Remark: "i" is a solution of the equation $x^2 = -1$. Because no real number satisfies this equation "i" is called an "imaginary number".

We can verify that $i^2 = -1$ is consistent with def.

Eg evaluate $(2+3i)(4+5i)$:

Foil.
$$= 2(4) + 2(5i) + 3i(4) + 3i(5i)$$
$$= 8 + 10i + 12i + 15i^2 \quad [i^2 = -1]$$
$$= 15i^2 + 22i + 8$$
$$= -15 + 22i + 8$$
$$= -7 + 22i$$

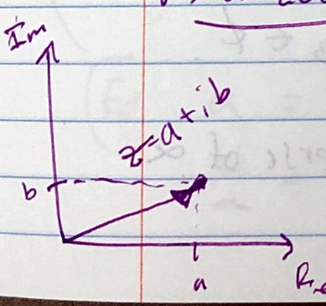
Notation: $z = a + ib$
$a = Re(z)$
$b = Im(z)$

Eg $Re(z) = 2$, $Im(z) = -1$ $\} \rightarrow z = 2 - i$

Visualization: $z = a + ib \in \mathbb{C}$ ("belongs to")

$z = a + ib$ can be seen as a coordinate of a point in 2D space

$z = (a, b)$



Properties of $\mathbb{C}$ $\qquad \alpha, \beta \in \mathbb{C}$

(i) · Commutativity ~~Comitivity~~

$\alpha + \beta = \beta + \alpha, \quad \alpha\beta = \beta\alpha \quad \overset{\text{"for all"}}{\forall} \quad \alpha, \beta \in \mathbb{C}$

Proof add. $\alpha = a + ib$

$\beta = c + id$

$\alpha + \beta = (a+ib) + (c+id) = (a+c) + i(b+d)$

$= (c+a) + i(d+b)$

$= (c+id) + (a+ib) = \beta + \alpha$ #

mult.

$\alpha\beta = (a+ib)(c+id) \neq (ac + ib(id) + ib(c) + a(id))$

$i^2 = -1 \qquad = -bd + i(bc + ad) + ac$

$= (ac - bd) + i(bc + ad)$

$\beta\alpha = (c+id)(a+ib) = (ca - db) + i(cb + da)$

$\Rightarrow \alpha\beta = \beta\alpha$

(ii) associativity

$(\alpha + \beta) + \lambda = \alpha + (\beta + \lambda) \quad \forall \ \alpha, \beta, \lambda \in \mathbb{C}$

$(\alpha\beta)\lambda = \alpha(\beta\lambda) \quad \forall \ \alpha, \beta, \lambda \in \mathbb{C}$

(iii) identities

$\lambda + 0 = \lambda$

$\lambda 1 = \lambda \qquad \forall \lambda \in \mathbb{C}$

(iv) addition & multiplication Inverse

$\underset{\text{(for any)}}{\forall} \alpha \in \mathbb{C}, \ \underset{\text{(there exists)}}{\exists} \underset{\text{(unique)}}{!} \ \beta \underset{\text{(belongs to)}}{\in} \mathbb{C} \ \underset{\text{(such that)}}{\text{s.t.}} \ \alpha + \beta = 0$

$\forall \alpha \in \mathbb{C} \quad \exists ! \beta \in \mathbb{C} \text{ s.t. } \alpha\beta = 1.$

$\alpha \neq 0$

(V) $\lambda(\alpha + \beta) = \lambda\alpha + \lambda\beta. \quad \forall \ \lambda, \alpha, \beta \in \mathbb{C}.$

Def. $\quad \alpha, \beta \in \mathbb{C}$

(i) $(-\alpha)$ denotes the additive inverse of α

$\alpha + (-\alpha) = 0$

Remark : $\alpha = a + ib$

$-\alpha = (-a) + i(-b)$

(ii) subtraction :

$\beta - \alpha = \beta + (-\alpha)$

(iii) for $\alpha \neq 0$

let $\frac{1}{\alpha}$ be the multiplicative inverse of α.

$\alpha\left(\frac{1}{\alpha}\right) = 1$

Remark : $\alpha = a + ib$, then

$$\frac{1}{\alpha} = \frac{a}{a^2+b^2} - \frac{b}{a^2+b^2} i$$

proof : WTS : $\alpha\left(\frac{1}{\alpha}\right) \overset{?}{=} 1$

$$\alpha\left(\frac{1}{\alpha}\right) = (a+ib)\left(\frac{a}{a^2+b^2} - \frac{b}{a^2+b^2} i\right) = \cdot\sim$$

$$\cdot\sim = \frac{a^2}{a^2+b^2} - \cancel{\frac{ab}{a^2+b^2}} i + \cancel{\frac{ab}{a^2+b^2}} i - \frac{b^2}{a^2+b^2}\underset{(-1)}{(i^2)}$$

$$= \frac{a^2}{a^2+b^2} + \frac{b^2}{a^2+b^2} = \frac{a^2+b^2}{a^2+b^2} = 1 \quad ✓$$

Notation: F stands for $\mathbb{C}$ or $\mathbb{R}$.

Field

- elements of F are called scalers.
- Defn. $\alpha \in F$, $m \in \mathbb{N}$

$$\alpha^m = \underbrace{\alpha \cdot \alpha \cdot \alpha \cdots \alpha}_{m\text{-times}}$$

one can show that

$$(\alpha^m)^n = \alpha^{mn} \quad \forall\, \alpha, \beta \in F$$

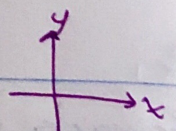
$$(\alpha\beta)^n = \alpha^n\beta^n \quad m, n \in \mathbb{N}.$$

Ex

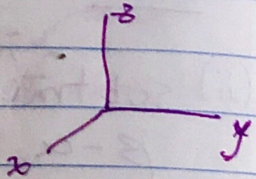
$\alpha = 3 + 4i$

$$\frac{1}{\alpha} = \frac{3}{25} - \frac{4}{25} i$$

(Eg) $\mathbb{R}^2 = \{(x,y) \mid x, y \in \mathbb{R}\} \rightarrow$



$\mathbb{R}^3 = \{(x,y,z) \mid x, y, z \in \mathbb{R}\} \rightarrow$



Def A list of length n:

$(x_1, x_2, \dots, x_n)$

two lists are equal iff they have the same length and same elements in the same order.

$x = (x_1, x_2, \dots, x_n)$

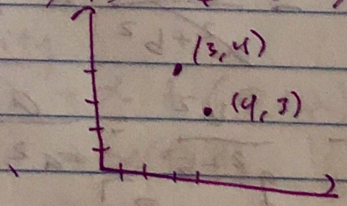
$y = (y_1, y_2, \dots, y_n)$

$x = y \iff x_i = y_i \ \forall\, i \in \{1, \dots, n\}$

(Eg) sets: both $\{3,4\}$ and $\{4,3\}$ are equal.

But

__lists__ $(3,4) \neq (4,3)$



$\mathbb{R}$

$\mathbb{R}^2 = \{(a,b) \mid a, b \in \mathbb{R}\}$

$\mathbb{R}^n = \{(x_1, x_2 \dots x_n) \mid (x_i) \in \mathbb{R}\}$

$\mathbb{C}^n = \{(x_1, x_2, \dots x_n) \mid x_i \in \mathbb{C}\}$

Def: $F^n := \left\{(x_1, x_2 \dots x_n) \mid \begin{matrix} x_i \in F \\ i \in \{1, 2, \dots n\} \end{matrix}\right\}$

(Eg): $\mathbb{C}^4 = \left\{(z_1, z_2, z_3, z_4) \mid \begin{matrix} z_i \in \mathbb{C} \\ i \in (1,2,3,4) \end{matrix}\right\}$

Def Addition in F^n; $x, y \in F^n$

$x = (x_1 \cdots x_n)$ $\quad x_i, y_i \in F$

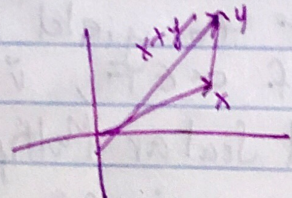
$y = (y_1 \cdots y_n)$

$x + y = (x_1 \cdots x_n) + (y_1 \cdots y_n)$

$:= (x_1 + y_1 \cdots x_n + y_n)$

Geometric Intuition:

$x, y \in \mathbb{R}^2$



Def Scalar Multiplication

$\lambda \in F$, $(x_1 \cdots x_n) \in F^n$

$\lambda(x_1 \cdots x_n) := (\lambda x_1, \cdots, \lambda x_2)$

Eg $\lambda = 2 - i$

$x \in \mathbb{C}^2$ s.t $x = (3-2i, 5i)$

Find $\lambda x = (2-i)(3-2i, 5i)$

$= ((2-i)(3-2i), (2-i)(5i)) =$

$6 - 7i + 2i^2 \quad 10i - 5i^2$

$6 - 7i - 2 \quad 10i + 5$

$(-7i + 4, 10i + 5)$

* HW? Ex 1A: 4, 5, 6, 9, 10, 11, 12, 13, 15, 16

Def Addition in $\mathbb{F}^n$; $x, y \in \mathbb{F}^n$

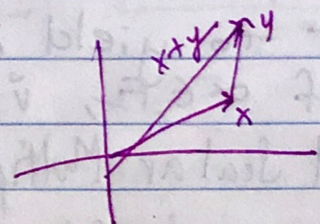
$x = (x_1 \cdots x_n)$ $\quad x_i, y_i \in \mathbb{F}$

$y = (y_1 \cdots y_n)$

$$x + y = (x_1 \cdots x_n) + (y_1 \cdots y_n)$$
$$:= (x_1 + y_1, \cdots x_n + y_n)$$

Geometric Intuition:

$x, y \in \mathbb{R}^2$



Def Scalar Multiplication

$\lambda \in \mathbb{F}$, $(x_1 \cdots x_n) \in \mathbb{F}^n$

$\lambda(x_1 \cdots x_n) := (\lambda x_1, \cdots, \lambda x_2)$

Eg $\lambda = 2 - i$

$x \in \mathbb{C}^2$ s.t $x = (3 - 2i, 5i)$

Find $\lambda x = (2-i)(3-2i, 5i)$

$= ((2-i)(3-2i), (2-i)(5i)) =$

$6 - 7i + 2i^2 \qquad 10i - 5i^2$

$6 - 7i - 2 \qquad 10i + 5$

$(-7i + 4, 10i + 5)$

* HW? Ex 1A: 4, 5, 6, 9, 10, 11, 12, 13, 15, 16