$\mathbb{R}$: Real numbers

Math 131

6/24/19 1A

week 1

Monday

Def. complex numbers;

- A complex number is a ordered pair (a, b)

$a, b \in \mathbb{R}$, $\alpha = a + ib$

- The set of all complex numbers: $\mathbb{C} := \{ a+ib \mid a, b \in \mathbb{R}$

Addition & Multiplication on $\mathbb{C}$:

$(a+ib) + (c+id) := (a+c) + i(b+d)$

$\in \mathbb{C}$ $\in \mathbb{C}$

$(a+ib)(c+id) := (ac-bd) + i(ad+bc)$

Remark: "i" is a soln of the eqth $x^2 = -1$. b/c no real # satisfies this eqth, "i" is called an "imaginary #".

We can verify that $i^2 = -1$ is consistent w/ def.

Ex. Evaluate $(2+3i)(4+5i)$

$8 + 10i + 12i + \underbrace{15i^2}_{15(-1)} = \boxed{-7 + 22i}$

$z = a + ib$: Notation

$a = Re(z)$

$b = Im(z)$

Ex.

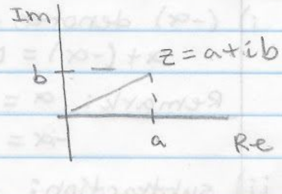
$\left. \begin{array}{l} Re(z) = 2 \\ Im(z) = -1 \end{array} \right\} z = 2 - i$

"belongs to"

Visualization: $z = a + ib \in \mathbb{C}$

$z = a+ib$ can be seen as a coordinate of a point in a two-dim space

$z = (a, b)$



Properties of $\mathbb{C}$: $\alpha, \beta \in \mathbb{C}$;

i) commutivity

forall

$\alpha + \beta = \beta + \alpha$, $\alpha\beta = \beta\alpha$ $\forall\, \alpha, \beta \in \mathbb{C}$

Proof; $\alpha = a + ib$ $\qquad \alpha+\beta = \beta+\alpha$

$\beta = c + id$

$\alpha+\beta = (a+ib)+(c+id) = (a+c) + i(b+d)$

$= (c+a) + i(d+b)$

$= (c+id)+(a+ib) = \beta+\alpha$

Proof; $\alpha = a+ib$ $\qquad \alpha\beta = \beta\alpha$

$\beta = c+id$

$\alpha\beta = (a+ib)(c+id) = (ac-bd)+i(ad+bc)$

$\beta\alpha = (c+id)(a+ib) = (ca-db)+i(cb+da)$

$\Rightarrow \alpha\beta = \beta\alpha$

ii) associativity

$(\alpha+\beta)+\lambda = \alpha+(\beta+\lambda) \quad \forall\, \alpha,\beta,\lambda \in \phi$

$(\alpha\beta)\lambda = \alpha(\beta\lambda) \qquad \forall\, \alpha,\beta,\lambda, \in \phi$

iii) identities

$\lambda + 0 = \lambda$

$\lambda 1 = \lambda$ $\qquad \forall\, \lambda\in\phi$

iv) additive & multiplication Inverse

$\forall\, \alpha\in\phi,\ \exists!\, \beta\in\phi$ s.t. $\alpha+\beta = 0$

for any — there exists — unique — belongs to — such that

$\forall\, \alpha\in\phi\ \exists!\, \beta\in\phi$ s.t. $\alpha\beta = 1.$

Distributive $\alpha\neq 0$

V) $\lambda(\alpha+\beta) = \lambda\alpha+\lambda\beta \quad \forall\, \lambda,\alpha,\beta\in\phi$

Def. $\alpha,\beta\in\phi$

i) $(-\alpha)$ denotes the additive inverse of α.

$\alpha+(-\alpha) = 0$

Remark: $\alpha = a+ib$

$-\alpha = (-a)+i(-b)$

ii) subtraction:

$\beta-\alpha = \beta+(-\alpha)$

iii) for $\alpha \neq 0$

let $1/\alpha$ be the multiplicative inverse of α

$\alpha\left(\frac{1}{\alpha}\right) = 1$

Remark: $\alpha = a + ib$, then $\frac{1}{\alpha} = \frac{a}{a^2+b^2} - \frac{b}{a^2+b^2} i$

Proof: WTS: $\alpha\left(\frac{1}{\alpha}\right) = 1$

$\alpha = a + ib$ $\quad \alpha\left(\frac{1}{\alpha}\right) = (a+ib)\left(\frac{a}{a^2+b^2} - \frac{b}{a^2+b^2} i\right)$

$\beta = c + id$

$$= \frac{a^2}{a^2+b^2} - \cancel{\frac{ab}{a^2+b^2} i} + \cancel{\frac{ab}{a^2+b^2} i} - \frac{b^2}{a^2+b^2} \underbrace{(i^2)}_{-1}$$

$$= \frac{a^2}{a^2+b^2} + \frac{b^2}{a^2+b^2} = \frac{a^2+b^2}{a^2+b^2} = 1$$

Notation: $\underset{\text{field}}{\underset{\uparrow}{F}}$ stands for $\mathbb{C}$ or $\mathbb{R}$.

- Elements of F are called scalars
- Def. $a \in F$, $m \in \mathbb{N}$

$$a^m = \underbrace{\alpha \cdot \alpha \cdot \alpha \cdots \alpha}_{m\text{-times}}$$

one can show that:

$(\alpha^m)^n = \alpha^{mn}$

$(\alpha\beta)^n = \alpha^n \beta^n \quad \forall \alpha, \beta \in F$

$m, n \in \mathbb{N}$

Ex. $\alpha = 3 + 4i$

$\frac{1}{\alpha} = \frac{3}{25} - \frac{4}{25} i$

Ex. $\mathbb{R}^2 = \{(x,y) \mid x, y \in \mathbb{R}\} \rightarrow$

$\mathbb{R}^3 = \{(x,y,z) \mid x, y, z, \in \mathbb{R}\} \rightarrow$

Def. A list of length n: $(x_1, x_2, \ldots, x_n)$

Two lists are equal if they have same length & the same elements in the same order

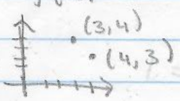
$x = (x_1, x_2, \ldots, x_n)$

$y = (y_1, y_2, \ldots, y_n)$ $\quad x = y \iff x_i = y_i \ \forall i \in \{1, \ldots, n\}$

Ex. sets: both $\{3,4\}$ & $\{4,3\}$ are equal

but

lists: $(3,4) \neq (4,3)$



$\mathbb{R}$

$\mathbb{R}^2 = \{(a,b) \mid a, b \in \mathbb{R}\}$

$\mathbb{R}^n = \{(x_1, x_2, \ldots, x_n) \mid x_i \in \mathbb{R}\}$

$\mathbb{C}^n = \{(x_1, x_2, \ldots, x_n) \mid x_i \in \mathbb{R}\}$

Def.

$F^n := \{(x_1, x_2, \ldots, x_n) \mid x_i \in F, \ i \in \{1, 2, \ldots, n\}\}$

Ex.

$\mathbb{C}^4 = \{(z_1, z_2, z_3, z_4) \mid z_i \in \mathbb{C}, \ i \in \{1, 2, 3, 4\}\}$

Def. Addition in F^n:

$x, y \in F^n$

$x = (x_1, x_2, \ldots, x_n) \quad x_i \ y_i \in F$

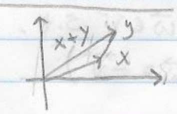
$y = (y_1, y_2, \ldots, y_n)$

$x + y = (x_1, \ldots, y_n) + (y_1, \ldots, y_n)$

$:= (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)$

Geometric Intuition :

$x, y \in \mathbb{R}^2$



Def Scalar multiplication

$\lambda \in F, (x_1, x_2, \ldots, x_n) \in F^n$

$\lambda(x_1, x_2, \ldots, x_n) := (\lambda x_1, \lambda x_2, \ldots, \lambda x_n)$

Ex. $\lambda = 2-i$, $x \in \mathbb{C}^2$ s.t. $x = (3-2i, 5i)$

Find $\lambda x = (2-i)(3-2i, 5i)$

$= ((2-i)(3-2i), (2-i)(5i))$

$= (6-4i-3i+2i^2, 10i-5i^2)$

$= (6-2-7i, 10i+5) = \boxed{(4-7i, 5+10i)}$

Ex. 1.A: 4, 5, 6, 9, 10, 11, 12, 13, 15, 16

6/25/19 1.B vector space

week 1

Tues.

$(V, +, \cdot)$

V → set of vectors

$+$ → addition

$\cdot$ → scalar multiplication

A vector space consists of a set of vectors V & a field F

- the vectors can be added to yield another vector in V:

  $\forall \vec{u}, \vec{v} \in V, \vec{u} + \vec{v} \in V$

- the scalar can be multiplied w/ the vector to yield a new vector in V:

  If $\alpha \in F$, $\vec{v} \in V$ then $\alpha \cdot \vec{v} \in V$

Addition & scalar multiplication must also satisfy the following axioms:

1) commutivity $\vec{u} + \vec{v} = \vec{v} + \vec{u}$ $\forall \vec{u}, \vec{v} \in V$

2) associativity $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$ $\forall \vec{u}, \vec{v}, \vec{w} \in V$

$(\alpha \beta)\vec{v} = \alpha(\beta \vec{v})$ $\forall \alpha, \beta \in F$