LECTURE 01
06/24/19

MATH 131

Section 1A

Def Complex numbers

- A complex number is an ordeal pair (a,b) $a,b \in \mathbb{R}$,
  $\alpha = a + ib$
- the set of all complex numbers $\mathbb{C} := \{a+ib \mid a,b \in \mathbb{R}\}$
- Addition and multiplication on $\mathbb{C}$:

$$\underbrace{(a+ib)}_{\in \mathbb{C}} + \underbrace{(c+id)}_{\in \mathbb{C}} := (a+c) + i(b+d)$$

$$(a+ib)(c+id) := (ac-bd) + i(ad+bc)$$

Remark: "i" is a solution of the equation
$x^2 = -1$ Because no real number satisfies this equation, "i" is called an "imaginary number"

we can verify that $i^2 = -1$ is consistant with def

Ex: evaluate $(2+3i)(4+5i)$:

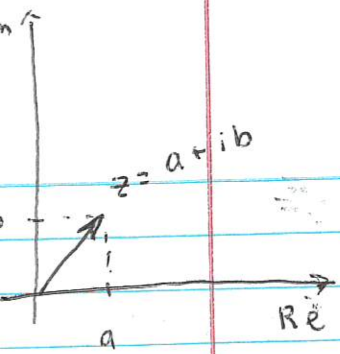
$2(4) + 2(5i) + (3i)(4) + (3i)(5i)$
$= 8 + 10i + 12i + 15i^2$ $\qquad i^2 = -1$
$= 8 + 22i - 15$
$= -7 + 22i$

Notation: $z = a + ib$
$a = \mathrm{Re}(z)$
$b = \mathrm{Im}(z)$

Ex: $\left.\begin{array}{l} \mathrm{Re}(z) = 2 \\ \mathrm{Im}(z) = -1 \end{array}\right\}$ $z = 2 - i$

Visualization: $z = a + ib \in \mathbb{C}$ $\qquad z = (a,b)$

$z = a+ib$ can be seen as a coordinate of a point in a two-dim space



properties of $\mathbb{C}$: $\alpha, \beta \in \mathbb{C}$:

(i) continuity

$\alpha + \beta = \beta + \alpha$, $\alpha\beta = \beta\alpha$ $\underset{\text{for all}}{\forall}$ $\alpha, \beta \in \mathbb{C}$

Proof $\alpha = a + ib$
$\beta = c + id$

$$\alpha + \beta = (a+ib) + (c+id) = (a+c) + i(b+d)$$
$$= (c+a) + i(d+b)$$
$$= (c+id+a+ib) = \beta + \alpha$$

$$\alpha\beta = (a+ib)(c+id) = $$
$$=$$
$$= (ac-bd) + i(ad+bc)$$

$$\beta\alpha = (c+id)(a+ib) = (ca-db) + i(cb+da)$$
$$\Rightarrow \alpha\beta = \beta\alpha$$

(ii) associativity

$(\alpha+\beta)+\lambda = \alpha + (\beta+\lambda)$ $\forall \alpha, \beta, \lambda \in \mathbb{C}$

$(\alpha\beta)\lambda = \alpha(\beta\lambda)$ $\forall \alpha, \beta, \lambda \in \mathbb{C}$

(iii) identities

$\lambda + 0 = \lambda$ $\forall \lambda \in \mathbb{C}$

$\lambda 1 = \lambda$

~~LECTURE 01~~

(iv) additive & multiplication Inverse

$\underset{\text{for any}}{\forall}\ \alpha \in \mathbb{C},\ \underset{\text{There exist}}{\exists}\ \underset{\text{unique}}{!}\ \beta \in \mathbb{C}\ \underset{\text{such that}}{\text{s.t.}}\quad \alpha + \beta = 0$

$\forall\, \alpha \in \mathbb{C}\ \ \exists!\ \beta \in \mathbb{C}\ \text{ s.t }\ \alpha\beta = 1$

$\alpha \neq 0$

(V) $\lambda(\alpha + \beta) = \lambda\alpha + \lambda\beta \qquad \forall\, \lambda, \alpha, \beta \in \mathbb{C}$

Def $\alpha, \beta \in \mathbb{C}$

(i) $(-\alpha)$ denotes the additive inverse of α

$$\alpha + (-\alpha) = 0$$

Remark $\alpha = a + ib$

$$-\alpha = (-a) + i(-b)$$

(ii) subtraction

$$\beta - \alpha = \beta + (-\alpha)$$

(iii) for $\alpha \neq 0$

Let $1/\alpha$ be the multiplicative inverse of α

$$\alpha\left(\frac{1}{\alpha}\right) = 1$$

Remark $\alpha = a + ib$, then

$$\frac{1}{\alpha} = \frac{a}{a^2+b^2} - \frac{b}{a^2+b^2}\, i$$

Proof: WTS: $\alpha\left(\frac{1}{\alpha}\right) \overset{?}{=} 1$

$$\alpha\left(\frac{1}{\alpha}\right) = (a+ib)\left(\frac{a}{a^2+b^2} - \frac{b}{a^2+b^2}\, i\right)$$

$$= \frac{a^2}{a^2+b^2} - \cancel{\frac{ab}{a^2+b^2}\, i} + \cancel{\frac{ab}{a^2+b^2}\, i} - \frac{b^2}{a^2+b^2}\underbrace{(i^2)}_{-1}$$

$$= \frac{a^2}{a^2+b^2} + \frac{b^2}{a^2+b^2} = \frac{a^2+b^2}{a^2+b^2} = 1$$

Notation: F stands for $\mathbb{C}$ or $\mathbb{R}$

field

- elements of F are called scalars
- Def $\alpha \in F$, $m \in \mathbb{N}$

$$\alpha^m = \underbrace{\alpha \cdot \alpha \cdot \alpha \cdots \alpha}_{m\text{ times}}$$

one can show that

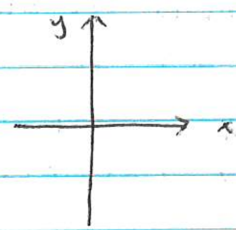
$$(\alpha^m)^n = \alpha^{m \cdot n}$$

$$(\alpha\beta)^n = \alpha^n \beta^n \qquad \forall \alpha, \beta \in F,\ m, n \in \mathbb{N}$$

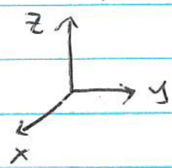
Ex: $\alpha = 3 + 4i$

$$\frac{1}{\alpha} = \frac{3}{25} - \frac{4}{25} i$$

Ex: $\mathbb{R}^2 = \{(x, y) \mid x, y \in \mathbb{R}\} \rightarrow$



$\mathbb{R}^3 = \{(x, y, z) \mid x, y, z \in \mathbb{R}\} \rightarrow$



Def A list of length n: $(x_1, x_2, \dots x_n)$

two lists are equal iff they have same length and same elements in the same order

$x = (x_1, x_2, \dots, x_n)$

$y = (y_1, y_2, \dots y_n)$

$x = y \Leftrightarrow x_i = y_i \quad \forall i \in \{1, \dots, n\}$

Ex: sets: both $\{3, 4\}$ and $\{4, 3\}$ are equal

lists: $(3, 4) \neq (4, 3)$

$\mathbb{R}$

$\hookrightarrow \mathbb{R}^2 = \{(a, b) \mid a + b \in \mathbb{R}\}$

$\mathbb{R}^n = \{(x_1, x_2, \dots, x_n) \mid x_i \in \mathbb{R}\}$

$\mathbb{C}^n = \{(x_1, x_2, \dots x_n) \mid x_i \in \mathbb{C}\}$

(3)

Def: $F^n := \{(x_1, x_2, \cdots, x_n) \mid x_i \in F,\ i \in \{1, 2, \cdots, n\}\}$

Ex:

$\mathbb{C}^4 = \{(z_1, z_2, z_3, z_4) \mid z_i \in \mathbb{C},\ i \in \{1, 2, 3, 4\}\}$

Def Addition in F^n:

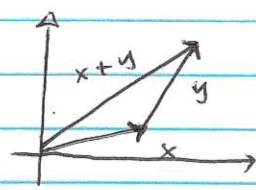
$x, y \in F^n$

$x = (x_1, x_2, \cdots, x_n)$ $x_i, y_i \in F$
$y = (y_1, y_2, \cdots, y_n)$

$x + y = (x_1, \cdots, x_n) + (y_1, \cdots y_n)$
$:= (x_1 + y_1, x_2 + y_2, \cdots x_n + y_n)$

Geometric intution:

$x, y \in \mathbb{R}^2$



Def scalar multiplication

$\lambda \in F$, $(x_1, x_2, \cdots x_n) \in F^n$
$\lambda (x_1, x_2, \cdots x_n) := (\lambda x_1, \lambda x_2, \cdots, \lambda x_n)$

Ex $\lambda = 2 - i$

$x \in \mathbb{C}^2$ s.t $x = (3 - 2i, 5i)$

Find $\lambda x = (2 - i)(3 - 2i, 5i)$

$= ((2-i)(3-2i), (2-i)(5i))$

$= (6 - 4i - 3i - 2, 10i + 5)$

$= (4 - 7i, 5 + 10i)$

Ex Sec 1.A

4, 5, 6, 9, 10, 11, 12, 13, 15, 16