Section 1A

Def. Complex numbers:

- A complex number is a ordered pair (a, b) $a, b \in \mathbb{R}$, $\alpha = a + ib$
- The set of all complex numbers: $\mathbb{C} := \{a+ib \mid a, b \in \mathbb{R}\}$
- Addition and Multiplication on $\mathbb{C}$:

$$\underbrace{(a+ib)}_{\in \mathbb{C}} + \underbrace{(c+id)}_{\in \mathbb{C}} := (a+c) + i(b+d)$$

$$(a+ib)(c+id) := (ac-bd) + i(ad+bc)$$

Remark: "i" is a solution of the equation $x^2 = -1$. Because no real number satisfies this equation, "i" is called an "imaginary number".

We can varify that $i^2 = -1$ is consistant with definition.

Ex. evaluate $(2+3i)(4+5i)$

$$(2+3i)(4+5i) = 2\times 4 + 2\times 5i + 12i + 15i^2 = 8 - 15 + 22i = -7 + 22i$$

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complex number $z = a + ib$

real part
$a = Re(z)$

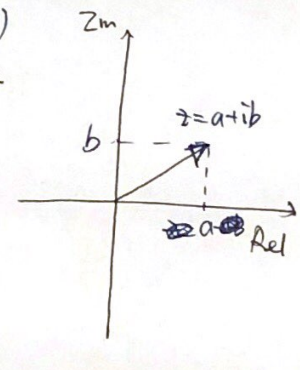
imaginary part
$b = Im(z)$

Ex. $\left.\begin{array}{l} Re(z) = 2 \\ Im(z) = -1 \end{array}\right\} \longrightarrow z = 2 - i$

"belongs to"

Visilation: $z = a + ib \in \mathbb{C} \longrightarrow z = (a, b)$

$z = a + ib$ can be seen as a coordinate of a point in a two-dimension space



properties of $\mathbb{C}$: $\alpha, \beta \in \mathbb{C}$;

(i) • Commutativity

$\alpha + \beta = \beta + \alpha$, $\alpha\beta = \beta\alpha$ $\overset{\text{for all}}{\forall} \alpha, \beta \in \mathbb{C}$

Proof: $\alpha = a + ib$, $\beta = c + id$

$\alpha + \beta = (a+ib) + (c+id) = (a+c) + i(b+d) = (c+a) + i(d+b)$
$= (c+id) + (a+ib) = \beta + \alpha$

$\alpha\beta = \beta\alpha$ $\forall \alpha, \beta \in \mathbb{C}$? $\alpha\beta = (a+ib)(c+id) = (ac - bd) + i(ad + bc)$ $\shortparallel$ ✓

~~$\beta\alpha = (\overset{ca}{ac} - \overset{db}{bd}) + i(bc + ad)$~~ $\beta\alpha = (c+id)(a+ib) = (ca - db) + i(cb + da)$

$\therefore \Longrightarrow \alpha\beta = \beta\alpha$

(ii) • associativity

$(\alpha + \beta) + \lambda = \alpha + (\beta + \lambda)$ $\forall \alpha, \beta, \lambda \in \mathbb{C}$

$(\alpha\beta)\lambda = \alpha(\beta\lambda)$ $\forall \alpha, \beta, \lambda \in \mathbb{C}$

(iii) • identities

$\lambda + 0 = \lambda$ $\forall \lambda \in \mathbb{C}$

$\lambda 1 = \lambda$

(iv) • additive & multiplication Inverse

$\underset{\text{for any}}{\forall} \alpha \in \mathbb{C}$, $\underset{\text{there exist}}{\exists} \underset{\text{unique}}{!}$ $\beta \in \mathbb{C}$ $\overset{\text{belongs to}}{}$ $\underset{\text{Such that}}{\text{s.t.}}$ $\alpha + \beta = 0$

$\forall \alpha \in \mathbb{C}$ $\exists! \beta \in \mathbb{C}$ s.t. $\alpha\beta = 1$

$\alpha \neq 0$

(V) • $\lambda(\alpha + \beta) = \lambda\alpha + \lambda\beta$. $\forall \lambda, \alpha, \beta \in \mathbb{C}$

Def. $\alpha, \beta \in \mathbb{C}$

(i) $(-\alpha)$ denotes the additive inverse of α.

$\alpha + (-\alpha) = 0$

Remark: $\alpha = a + ib$

$-\alpha = (-a) + i(-b)$

(ii) Substitution: $\beta - \alpha = \beta + (-\alpha)$

(iii) for $\alpha \neq 0$

Let $1/\alpha$ be the multiplicative inverse of α.

$\alpha \left(\frac{1}{\alpha}\right) = 1$

Remark: $\alpha = a + ib$, then $\frac{1}{\alpha} = \frac{a}{a^2+b^2} - \frac{b}{a^2+b^2} i$

proof: Want to show: $\alpha\left(\frac{1}{\alpha}\right) \overset{?}{=} 1$

$\alpha\left(\frac{1}{\alpha}\right) = (a+ib) \times \left(\frac{a}{a^2+b^2} - \frac{b}{a^2+b^2} i\right)$ $\qquad (a-bi)(a+bi) = a^2 - (bi)^2 = a^2 + b^2.$

$\cancel{= \frac{(a+ib)(a^2+b^2)}{a^2+b^2} \times \frac{a-bi}{a^2+b^2}}$

$= a \times \frac{a}{a^2+b^2} - \frac{ab}{a^2+b^2} i + \frac{ab}{a^2+b^2} i - \frac{b^2}{a^2+b^2} \underbrace{i^2}_{= -1}$

$= \frac{a^2}{a^2+b^2} + \frac{b^2}{a^2+b^2} = \frac{a^2+b^2}{a^2+b^2} = 1$

Notation: $\underset{\text{field}}{F}$ stands for $\mathbb{C}$ or $\mathbb{R}$

- Elements of F are called scalers.
- Def. $\alpha \in F$, $m \in \mathbb{N}$ $\quad \alpha^m = \underbrace{\alpha \cdot \alpha \cdot \alpha \cdots \alpha}_{m\text{-times}}$

one can show that: $(\alpha^m)^n = \alpha^{mn}$ $\quad \forall \alpha, \beta \in F$

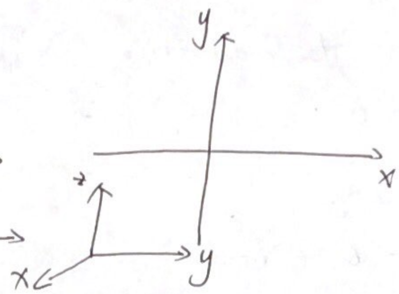
$(\alpha\beta)^n = \alpha^n \beta^n$. $\quad m, n \in \mathbb{N}$

Ex: $\alpha = 3 + 4i$

$$\frac{1}{\alpha} = \frac{3}{25} - \frac{4}{25} i.$$

Ex: $\mathbb{R}^2 = \{(x, y) \mid x, y \in \mathbb{R}\} \longrightarrow$

$\mathbb{R}^3 = \{(x, y, z) \mid x, y, z \in \mathbb{R}\} \longrightarrow$



Def. A list of length n: $(x_1, x_2, \cdots, x_n)$

Two lists are equal iff they have same length and the same elements in the same order.

if and only if

$x = (x_1, x_2, \cdots, x_n)$

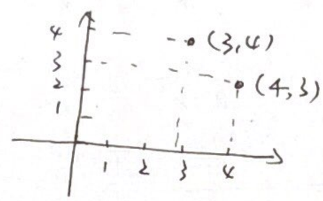
$y = (y_1, y_2, \cdots, y_n) \qquad x = y \iff x_i = y_i \quad \forall i \in \{1, \ldots, n\}$

Ex: Sets: both $\{3, 4\}$ and $\{4, 3\}$ are equal.

But,

lists: $(3, 4) \neq (4, 3)$

因为 order 不同.



$\mathbb{R} \longrightarrow \mathbb{R}^2 = \{(a, b) \mid a, b \in \mathbb{R}\}$

$\longrightarrow \mathbb{R}^n = \{(x_1, x_2, x_3, \ldots, x_n) \mid x_i \in \mathbb{R}\}$

$\mathbb{C}^n = \{(x_1, x_2, \cdots, x_n) \mid x_i \in \mathbb{C}\}$

~~Ex~~: $\cancel{\mathbb{C}^4 = \{(z_1, z_2, z_3, z_4) \mid z_i \in \mathbb{C},\ i \in \{1, 2, 3, 4\}\}}$

Def. $F^n := \left\{ (x_1, x_2, \cdots, x_n) \;\middle|\; \begin{array}{l} x_i \in F \\ i \in \{1, 2, \cdots, n\} \end{array} \right\}$

Ex: $\mathbb{C}^4 = \left\{ (z_1, z_2, z_3, z_4) \;\middle|\; \begin{array}{l} z_i \in \mathbb{C} \\ i \in \{1, 2, 3, 4\} \end{array} \right\}$

Def. Addition in F^n:

$x, y \in F^n$

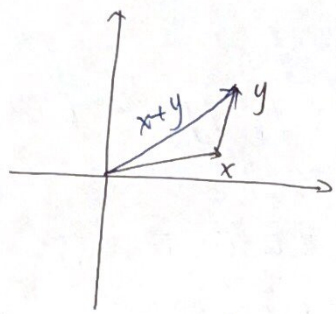
$x = (x_1, x_2, \dots, x_n)$ $\quad x_i, y_i \in F$ when $i \in \{1, \dots, n\}$

$y = (y_1, y_2, \dots, y_n)$

$x+y = (x_1, \dots x_n) + (y_1, \dots y_n)$
$= (x_1+y_1, x_2+y_2, \dots, x_n+y_n)$

Geometric Intution:

$x, y \in \mathbb{R}^2$



Def. Scalar Multiplication

$\lambda \in F$, $(x_1, x_2, \dots, x_n) \in F^n$,

$\lambda(x_1, x_2, \dots, x_n) = (\lambda x_1, \lambda x_2, \dots, \lambda x_n)$

Ex. $\lambda = 2-i$, $\quad x \in \mathbb{C}^2$ s.t. $x = (3-2i, 5i)$

Find $\lambda x = (2-i)(3-2i, 5i)$

$= ((2-i)(3-2i), (2-i)(5i)) = (6-4i-3i-2, 10i+5)$

$= (4-7i, 5+10i)$