1.B Vector Space $(V, +, \cdot)$

V: set of vectors, $+$: addition, $\cdot$: scalar multiplication.

A vector space consists of a set of vectors V and a field F.

- the vectors can be added to yield another vector in V:
  $\forall \vec{u}, \vec{v} \in V, \quad \vec{u} + \vec{v} \in V$
- the scalar can be multiplied with the vector to yield a new vector in V.
  If $\alpha \in F$, $\vec{v} \in V$, then $\alpha \cdot \vec{v} \in V$.

Addition and scalar multiplication must also satisfy the following axioms:

1). Communtivities: $\vec{u} + \vec{v} = \vec{v} + \vec{u} \quad \forall \vec{u}, \vec{v} \in V$

2). Associativities: $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w}) \quad \forall \vec{u}, \vec{v}, \vec{w} \in V$

$(\alpha\beta)\vec{v} = \alpha(\beta\vec{v}) \quad \forall \alpha, \beta \in F$

3). Additive Identity: $\exists \vec{0} \in V$ s.t. $\vec{0} + \vec{v} = \vec{v}, \quad \forall \vec{v} \in V$

4). Additive Inverse: $\forall \vec{v} \in V, \exists \vec{w} \in V$ s.t. $\vec{v} + \vec{w} = \vec{0}$

(Note: $\vec{w} = -\vec{v}$)

5). Multiplication Identity: $1 \cdot \vec{v} = \vec{v} \quad \forall \vec{v} \in V$

6). Distributive property: $\alpha(\vec{u} + \vec{v}) = \alpha\vec{u} + \alpha\vec{v}$

$(\alpha + \beta)\vec{v} = \alpha\vec{v} + \beta\vec{v}$

$\forall \vec{u}, \vec{v} \in V, \ \forall \alpha, \beta \in F$

Remarks: 1). Elements of a vector space are called vector.

2). We will say that V is a vector space over F instead of saying simply that V is a vector space.

Ex: $\mathbb{R}, \mathbb{R}^2, \cdots, \mathbb{R}^n$ are vector spaces

$\mathbb{C}, \mathbb{C}^2, \cdots, \mathbb{C}^n$ - - - - - - -

Def. • A vector space over $\mathbb{R}$ (i.e. $F=\mathbb{R}$) is called a real vector space.

• A vector space over $\mathbb{C}$ (i.e. $F=\mathbb{C}$) is called a complex vector space.

Ex. (Recall : $F=\mathbb{R}$ or $\mathbb{C}$)

$V=F^n = \{(x_1, x_2, \cdots, x_n) \mid x_1, x_2, \cdots, x_n \in F\}$

Addition: $\underbrace{(x_1, x_2, \cdots, x_n)}_{\in F^n} + \underbrace{(y_1, y_2, \cdots, y_n)}_{\in F^n} = \underbrace{(x_1+y_1, \cdots, x_n+y_n)}_{\in F^n}$

Scalar multiplication: $\alpha \in F$

$$\alpha \underbrace{(x_1, x_2, \cdots, x_n)}_{\in F^n} = \underbrace{(\alpha x_1, \alpha x_2, \cdots, \alpha x_n)}_{\in F^n}$$

Ex: $\mathbb{R}^3 = \{(x_1, x_2, x_3) \mid x_i \in \mathbb{R}\}$ over $\mathbb{R}$

$F=\mathbb{R}$, $n=3$ $\quad + : (x_1, x_2, x_3) + (y_1, y_2, y_3) = (x_1+y_1, x_2+y_2, x_3+y_3)$

$\cdot : \alpha(x_1, x_2, x_3) = (\alpha x_1, \alpha x_2, \alpha x_3) \in \mathbb{R}^3$

Ex: $F^S = \{f : S \to F\}$ = set of functions from set S to field F.

• Addition, $f, g \in F^S$.

$$(f+g)(x) = f(x) + g(x) \quad \forall x \in S$$

- Scalar multiplication:

$$(\alpha f)(x) = \alpha f(x) \quad \forall \alpha \in F \quad \forall f \in F^S$$

F^S is a vector space over F.

Properties vector space:

I) A vector space has a unique additive identity.

Recall: additive Id : $0 + \vec{v} = \vec{v}$

Pf: Suppose there are two additive Ids: 0 & $0'$

$\vec{0} + \vec{v} = \vec{v}$ and

$\vec{0'} + \vec{v} = \vec{v}$

$0 + 0' = 0'$ → 0 is additive Id

'' commutivity

$0' + 0 = 0$ → $0'$ is also additive Id

''
0 ⇒ $0' = 0$

II) Unique additive inverse.

pf: V vectorspace $\vec{v} \in V$, suppose there are two additive inverse.

i.e. $\exists\, W, W' \in V$ s.t.

$V + W = 0$
⇒ WTS $W = W'$
$V + W' = 0$

$$W = W + 0 = W + (V + W') = (W + V) + W' = W' \Rightarrow W = W'$$

Note:
- $-\vec{v}$ → additive inverse of $\vec{v}$
- $\vec{w} - \vec{v} = \vec{w} + (-\vec{v})$

III) The number 0 times a vector

scalar $0 \cdot \vec{v} = \vec{0}$

vector   vector (additive Id)

Pf: $0\vec{v} = (0+0)\vec{v} = 0\vec{v} + 0\vec{v} \Rightarrow 0\vec{v} = 0$

$(\alpha+\beta)\vec{v} = \alpha\vec{v} + \beta\vec{v}$ distributive property.

IV) $\alpha\vec{0} = \vec{0} \quad \forall \alpha \in F$

Pf: $\alpha\vec{0} = \alpha(\vec{0}+\vec{0}) = \alpha\vec{0} + \alpha\vec{0} \Rightarrow \alpha\vec{0} = 0$

V) The number (-1) times a vector

$(-1)\vec{v} = -\vec{v}$ for any $\vec{v} \in V$.

$\vec{v} + \vec{w} = 0$ unique $w = -v$.

$\vec{v} + (-\vec{v}) = 0$

$\vec{v} + (-1)\vec{v} = \vec{v} \cdot (1-1) = \vec{v} \cdot 0 = 0$

Ex. 1.B.

2) Suppose $a \in F$, $v \in V$ and $a\vec{v} = \vec{0}$ prove that $a=0$ or $\vec{v} = \vec{0}$

$a \cdot \vec{v} = (a+0)\vec{v} = a\vec{v} + 0\vec{v} \quad \because 0\vec{v} = 0 = a\vec{v}$

$\therefore a = 0$ or $\vec{v} = \vec{0}$

If $a = 0$ ✓

If $a \neq 0$, so $\frac{1}{a}$ is well-defined

$\frac{1}{a}(a\vec{v}) = \frac{1}{a} \cdot \vec{0}$

$\vec{v} = \vec{0}$