LECTURE 02

06/25/19

MATH 131

Section 1B Vector Space $(V, +, \cdot)$

(set of vectors, addition, scalar multiplication)

A vectorspace consists of a set of vectors $\vec{v}$ and a field F

Recall: F = either $\mathbb{R}$ or $\mathbb{C}$

- the vectors can be added to yield another vector in V
  $\forall \vec{u}, \vec{v} \in V$, $\vec{u}+\vec{v} \in V$
- the scalar can be multiplied with the vector to yield a new vector in V:
  If $\alpha \in F$, $\vec{v} \in V$ then $\alpha \cdot \vec{v} \in V$

Addition and scalar multiplication must also satisfy the following axions

(1) Commutivity $\vec{u}+\vec{v} = \vec{v}+\vec{u}$ $\forall \vec{u}, \vec{v} \in V$

(2) Associativity $(\vec{u}+\vec{v})+\vec{w} = \vec{u}+(\vec{v}+\vec{w})$ $\forall \vec{u}, \vec{v}, \vec{w} \in V$
$(\alpha\beta)\vec{v} = \alpha(\beta\vec{v})$ $\forall \alpha, \beta \in F$

(3) Additive Identity, $\exists \vec{0} \in V$ s.t. $\vec{0}+\vec{v} = \vec{v}$ $\forall \vec{v} \in V$

(4) Additive Inverse, $\forall \vec{v} \in V$, $\exists \vec{w} \in V$ s.t. $\vec{v}+\vec{w} = \vec{0}$
(notation: $\vec{w} = -\vec{v}$)

(5) Multiplication Identity $1\vec{v} = \vec{v}$ $\forall \vec{v} \in V$

(6) Distributive property $\alpha(\vec{u}+(\vec{v})) = \alpha\vec{u} + \alpha\vec{v}$ $\forall \vec{u}, \vec{v} \in V$
$(\alpha+\beta)\vec{v} = \alpha\vec{v} + \beta\vec{v}$ $\alpha, \beta \in F$

Remarks: (1) Elements of a vector space are called vectors
(2) We will say that V is a vector space over F instead of saying simply the V is a vector space

Ex: $\mathbb{R}, \mathbb{R}^2, \ldots \mathbb{R}^n$ are vector spaces
$\mathbb{C}, \mathbb{C}^2, \ldots \mathbb{C}^n$ are vector spaces

Def A vector space over $\mathbb{R}$ (i.e: $F = \mathbb{R}$) is called a real vector space

- A vector space over $\mathbb{C}$ (i.e. $F = \mathbb{C}$) is called a complex vector space

Ex: $V = F^n := \{(x_1, x_2, \dots x_n) \mid x_1, x_2 \dots x_n \in F\}$ Recall: $F = \mathbb{R}$ or $\mathbb{C}$

Addition: $\underbrace{(x_1, x_2, \dots x_n)}_{F^n} + \underbrace{(y_1, y_2, \dots y_n)}_{\in F^n} := \underbrace{(x_1+y_1, x_2+y_2, \dots, x_n+y_n)}_{\in F^n}$

Scalar multiplication

$\alpha \in F$

$\alpha(x_1, x_2, \dots, x_n) := (\alpha x_1, \alpha x_2, \dots \alpha x_n) \in F^n$

with these Def $(F^n, +, \cdot)$ becomes a vector space over F

Ex: $\mathbb{R}^3 = \{(x_1, x_2, x_3) \mid x_i \in \mathbb{R}\}$

$F = \mathbb{R}$

$n = 3$

$\underbrace{(x_1, x_2, x_3)}_{\text{vector in } \mathbb{R}^3} + (y_1, y_2, y_3) = \overbrace{(x_1+y_1, x_2+y_2, x_3+y_3)}^{\text{vector in } \mathbb{R}3}$

$\cdot : \underset{\text{Scalar}}{\alpha} \underbrace{(x_1, x_2, x_3)}_{\text{vector}} := \underbrace{(\alpha x_1, \alpha x_2, \alpha x_3)}_{\text{vector}} \in \mathbb{R}^3.$

Ex: $V = F^S = \{f : S \to F\}$ = set of functions from set S to field F

- Addition $f, g \in F^S$,
  $(f+g)(x) = f(x) + g(x) \quad \forall x \in S$
- Scalar Multiplication $(\alpha f)(x) = \alpha f(x) \quad \forall \alpha \in F$
  $\forall f \in F^S$

F^S is a vector space over F.

Properties vector space:

(I) A vector space has a unique additive Identity

Recall: additive Id: $0 + \vec{v} = \vec{v}$

proof: suppose there are two additive Ids: 0 & $0'$

$\vec{0} + \vec{v} = \vec{v}$

$\vec{0}' + \vec{v} = \vec{v}$

$0 + 0' = 0'$ $\quad$ 0 is additive Id

$0' + 0 = 0$ $\quad$ $0'$ is also additive Id

$\Rightarrow 0' = 0$

(II) unique additive inverse

proof $\quad V$ vectorspace $\vec{v} \in V$, suppose there are two additive inverse

i.e. $\exists\ w, w' \in V$ s.t.

$$v + w = 0 \qquad \xRightarrow{\text{WTS}} \quad w = w' \, ?$$
$$v + w' = 0$$

$$w = w + 0 = w + (v + w') = (\underbrace{w + v}_{0}) + w' = w'$$

$$\Rightarrow w = w'$$

Notation

- $-\vec{v} \longrightarrow$ additive inverse of $\vec{v}$
- $\vec{w} - \vec{v} = \vec{w} + (-\vec{v})$

(III) The number 0 times a vector

$$0\vec{v} = \vec{0}$$

Scalar times vector equal to vector additive identities

proof: $\quad 0\vec{v} = (0+0)\vec{v} = 0\vec{v} + 0\vec{v} \Rightarrow 0\vec{v} = 0$

$(\alpha + \beta)\vec{v} = \alpha\vec{v} + \beta\vec{v}$ $\quad$ distributive property

(IV) $\alpha\vec{0} = \vec{0} \quad \forall \alpha \in F$

proof $\quad \alpha\vec{0} = \alpha(\vec{0} + \vec{0}) = \cancel{\alpha\vec{0}} + \alpha\vec{0} \Rightarrow \alpha\vec{0} = \vec{0}$

(V) The number (-1) times a vector $(-1)\vec{v} = -v$ for any $\vec{v} \in V$

$$\vec{v} + \vec{w} = 0 \text{ unique}$$
$$w = -v$$

$v + (-v) = 0$

$\vec{v} + (-1)\vec{v} = 0\,?$

$\hookrightarrow 1\vec{v} + (-1)\vec{v} = (1 + (-1))\vec{v} = 0\vec{v} = \vec{0}$

Ex 1.B.

(2) Suppose $a \in F$, $v \in V$ and $a\vec{v} = \vec{0}$ proove that $a=0$ or $\vec{v}=0$

$ab=0 \rightarrow a=0$ or $b=0$

If $a=0$

what if $a \neq 0 \overset{?}{\Rightarrow} \vec{v}=\vec{0}$

$a \neq 0$ so $\frac{1}{a}$ is well defined

$$\frac{1}{a}(a\vec{v}) = \left(\frac{1}{a}\vec{0}\right)$$

$$\left(\cancel{\frac{1}{a}a}\right)\vec{v} = \vec{0}$$

$$\vec{v} = \vec{0}.$$