1.B. Vector Space

$(V, +, \cdot)$

Set of vectors — addition — scalar multiplication

Recall, F = either $\mathbb{R}$ or $\mathbb{C}$

A Vectorspace consists of a set of Vectors V and a field F.

- the vectors can be added to yield another vector in V:

$\forall \vec{u}, \vec{v} \in V, \ \vec{u} + \vec{v} \in V$

- the scalar can be multiplied with the vector to yield a new vector in V:

If $\alpha \in F$, $\vec{v} \in V$ then $\alpha \cdot \vec{v} \in V$.

Addition and scalar multiplication must also satisfy the following axioms:

(1) Commutivity : $\vec{u} + \vec{v} = \vec{v} + \vec{u} \quad \forall \vec{u}, \vec{v} \in V$

(2) Associativity : $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w}) \quad \forall \vec{u}, \vec{v}, \vec{w} \in V$

$(\alpha\beta)\vec{v} = \alpha(\beta\vec{v}) \quad \forall \alpha, \beta \in F$

(3) additive Identity : $\exists \vec{0} \in V$ s.t. $\vec{0} + \vec{v} = \vec{v} \quad \forall \vec{v} \in V$

(4) additive Inverse : $\forall \vec{v} \in V$, $\exists \vec{w} \in V$ s.t. $\vec{v} + \vec{w} = \vec{0}$ (notation: $\vec{w} = -\vec{v}$)

(5) Multiplication Identity : $1\vec{v} = \vec{v} \quad \forall \vec{v} \in V$

(6) Distributive Property : $\alpha(\vec{u} + \vec{v}) = \alpha\vec{u} + \alpha\vec{v} \quad \forall \vec{u}, \vec{v} \in V$

$(\alpha + \beta)\vec{v} = \alpha\vec{v} + \beta\vec{v} \quad \alpha, \beta \in F$

<u>Remarks</u>: (1) Elements of a vector space are called vector.

(2) We will say that V is a vector space over F instead of saying simply the V is a vector space.

<u>Ex</u>. $\mathbb{R}, \mathbb{R}^2, \dots, \mathbb{R}^n$ are vector spaces.

$\mathbb{C}, \mathbb{C}^2, \dots, \mathbb{C}^n$ are vector spaces.

Def. • A vector space over $\mathbb{R}$ (i.e.: $F=\mathbb{R}$) is called a real vector space.

• A vector space over $\mathbb{C}$ (i.e.: $F=\mathbb{C}$) is called a complex vector space.

Recall: $F=\mathbb{R}$ or $\mathbb{C}$

Ex. $V=F^n := \{(x_1, x_2, \dots, x_n) \mid x_1, x_2, \dots, x_n \in F\}$

Addition: $\underbrace{(x_1, x_2, \dots, x_n)}_{F^n} + \underbrace{(y_1, y_2, \dots, y_n)}_{\in F^n} := \underbrace{(x_1+y_1, x_2+y_2, \dots, x_n+y_n)}_{\in F^n}$

Scalar Multiplication: $\alpha \in F$

$$\alpha(x_1, x_2, \dots, x_n) := (\alpha x_1, \alpha x_2, \dots, \alpha x_n) \in F^n.$$

With these Def. $(F^n, +; \cdot)$ becomes a vector space over F.

Ex. $\mathbb{R}^3 = \{(x_1, x_2, x_3) \mid x_i \in \mathbb{R}\}$ over $\mathbb{R}$

$F=\mathbb{R}$, $n=3$: $+$: $\underbrace{(x_1, x_2, x_3)}_{\text{Vector in } \mathbb{R}^3} + (y_1, y_2, y_3) = \underbrace{(x_1+y_1, x_2+y_2, x_3+y_3)}_{\text{Vector in } \mathbb{R}^3}$

$\cdot$: $\underset{\text{Scalar}}{\alpha}\underbrace{(x_1, x_2, x_3)}_{\text{Vector}} = \underbrace{(\alpha x_1, \alpha x_2, \alpha x_3)}_{\text{Vector}} \in \mathbb{R}^3$

Ex: $V=F^S = \{f: S \longrightarrow F\}$ = Set of functions from set S to field F

• Addition: $f, g \in F^S$, $(f+g)(x) = f(x) + g(x) \quad \forall x \in S$

• Scalar Multiplication: $(\alpha f)(x) = \alpha f(x) \quad \forall \alpha \in F \quad \forall f \in F^S$

F^S is a vector space over F.

Properties Vector space:

I) A vector space has a unique additive Identity.

Recall: additive Identity: $0 + \vec{v} = \vec{v}$

Proof: Suppose there are two additive Identities: 0 & $0'$

$\vec{0} + \vec{v} = \vec{v}$ and $\vec{0}' + \vec{v} = \vec{v}$

$0 + 0' = 0'$ → 0 is additive Identity.

$\|$ commutativity

$0' + 0$

$\|$ → $0'$ is also additive Identity.

0

$\Longrightarrow 0' = 0$ ✓

Recall: (4) $\forall v \in V \quad \exists w$ s.t. $\vec{v} + \vec{w} = 0$ Notice ~~Not~~ $\boxed{-\vec{v} = \vec{w}}$

II) Unique additive Inverse

Proof. V vectorspace, $\vec{v} \in V$, suppose there are two additive inverse

i.e. $\exists w, w' \in V$ s.t. $v + w = 0$, $v + w' = 0$ $\xrightarrow{\text{Want To Show}}$ $w = w'$?

$w = w + 0 = w + (v + w') = \underbrace{(w + v)}_{0} + w' = w'$

$\Longrightarrow \boxed{w = w'}$

Notation:
- $-\vec{v}$ → additive inverse of $\vec{v}$
- $\vec{w} - \vec{v} = \vec{w} + (-\vec{v})$

Properties vector space :

III) The number 0 times a vector

scalar $0\vec{v} = \vec{0}$ vector (additive Identity)

vector

Proof: $\cancel{0\vec{v}} = (0+0)\vec{v} = \cancel{0\vec{v}} + 0\vec{v} \Longrightarrow \boxed{0\vec{v} = 0}$

$(\alpha+\beta)\vec{v} = \alpha\vec{v} + \beta\vec{v}$ distributive property

IV) $\alpha\vec{0} = \vec{0} \quad \forall \alpha \in F$

Proof. $\alpha\vec{0} = \alpha(\vec{0}+\vec{0}) = \cancel{\alpha\vec{0}} + \alpha\vec{0} \Longrightarrow \boxed{\alpha\vec{0} = \vec{0}}$

V) The number (−1) times a vector

$(-1)\vec{v} = -\vec{v}$ for any $\vec{v} \in V$

Recall. unique
$\vec{v} + \vec{w} = 0$
$\vec{w} = -\vec{v}$

Proof. want to show: $v + (-v) = 0$

$\vec{v} + (-1)\vec{v} = 0$?

Recall: $\alpha\vec{v} + \beta\vec{v} = (\alpha+\beta)\vec{v}$

$(1)\vec{v} + (-1)\vec{v} = (1+(-1))\vec{v} = 0\cdot\vec{v} = \vec{0}$ ✓

Ex. 1.B.

(2) Suppose $a \in F$, $v \in V$ and $a\vec{v} = \vec{0}$. Prove that $a=0$ or $\vec{v} = \vec{0}$.

Soln. Want to show $ab = 0 \longrightarrow a=0$ or $b=0$

If $a=0$ ✓

what if $a \neq 0 \overset{?}{\Longrightarrow} \vec{v} = \vec{0}$

$a \neq 0$ so $\frac{1}{a}$ is well-defined

$\frac{1}{a}(a\vec{v}) = (\frac{1}{a}\vec{0})$

$(\frac{1}{a}\cdot a)\vec{v} = \vec{0}$

$\boxed{\vec{v} = \vec{0}}$ ✓