5. (-1) times a vector

$(-1)\vec{v} = -v$ for any $\vec{v} \in V$ $\qquad$ $\vec{v} + \vec{w} = 0$ unique

$w = -v$

$v + (-v) = 0$

$\vec{v} + (-1)v = 0$?

$1\vec{v} + (-1)\vec{v} = (1 + (-1))\vec{v}$

$= 0\vec{v} = \vec{0}$ ✓

$a\vec{v} + B\vec{v} = (a+B)\vec{v}$

$\overset{\shortparallel}{1} \quad \overset{\shortparallel}{-1}$

Exercise 1.B

2. suppose $a \in F$, vector $\in V$ and $a\vec{v} = \vec{0}$

prove that $a = 0$ or $\vec{v} = \vec{0}$

given $a\vec{v} = \vec{0}$

prop 3 states $0 \cdot \vec{v} = \vec{0}$

a could be 0

or $\vec{v}$ could be $\vec{0}$ and $a\vec{0} = \vec{0}$

$\boxed{6/26}$ 1.C $\boxed{\text{Sub Spaces}}$

$\boxed{1.32}$ $\boxed{\text{Def}}$ Let V = vector Space

Let U = subset of V

U = subspace of V IFF U is a vector space with add. & multi. scalar

$\boxed{1.34}$ $\boxed{\text{Conditions of subspace}}$

1. Add. Identity $0 \in U$

☆ 2. Closed under add. : $u, w \in U$ implies $u + w \in U$

3. Closed under multi : $a \in F$ and $u \in U$ implies $au \in U$.

Ex) $\boxed{1.35}$ a) If $b \in F$, then $\{(x_1, x_2, x_3, x_4) \in F^4 : x_3 = 5x_4 + b\}$ is a subspace if $b = 0$

Add Identi. Yes $\quad 0 = 5(0) + b = \boxed{0 = b}$

Closed under add.

$(x_1, x_2, x_3, x_4), (y_1, y_2, y_3, y_4) \in U_1$

$x_3 = 5x_4 + b \quad y_3 = 5y_4 + b$

$x_3 + y_3 = 5x_4 + b + 5y_4 + b = 5(x_4 + y_4) + 2b$

if $b = 0$, $x_3 = 5x_4$ & $y_3 = 5y_4$

$x_3 + y_3 = 5(x_4 + y_4)$

if $b = 0$ then,

$(x_1, x_2, x_3, x_4) + (y_1, y_2, y_3, y_4) = (x_1 + y_1, x_2 + y_2, x_3 + y_3, x_4 + y_4) \in U_1$

U_1 is closed under add.

Closed under multi.

$a \in F$ & $(x_1, x_2, x_3, x_4) \in U_1$

$a x_3 = 5a x_4 + ab$

if $b = 0$, $x_3 = 5x_4$ and $a x_3 = 5a x_4$

U_1 is closed under scalar multi.

U_1 is a subspace of V.

Sums of subspace

$\boxed{1.36}$ Definition: Sum of subsets

Let $U_1, \ldots, U_m$ be subsets of V. The sum of $U_1 \ldots U_m$ is denoted $U_1 + \ldots + U_m$ and is the set of all possible sums of elements of $U_1 \ldots U_m$

In other words:

$U_1 + \ldots U_m = \{u_1 + \ldots + u_m : u_1 \in U_1 \ldots U_m\}$

$\boxed{1.37}$ Example

$U = \{(x, 0, 0) \in F^3 : x \in F\}$

$W = \{(0, y, 0) \in F^3 : y \in F\}$

$U + W = \{(x, y, 0) : x, y \in F\}$

$(x, 0, 0) + (0, y, 0) \in U + W$ ✓

$= (x, y, 0) \in U + W$

$\boxed{1.39}$ ☆☆☆ Sum of subspaces is the smallest containing subspace

Suppose $U_1 \ldots U_m$ are subspaces of V prove that $U_1 + \ldots + U_m$ is the smallest subspace of V

Proof: Firsts claim $U_1 + \ldots + U_m$ is a subspace of V

Since $U_1, \ldots, U_m$ are subspaces of V, we have:

① Add. Identi: $0 \in U_1, \ldots, 0 \in U_m$

② Closed under add. $v_1, w_1 \in U_1, \ldots, v_m, w_m \in U_m$ implying $v_1 + w_1 \in U_1, \ldots, v_m + w_m \in U_m$

③ Closed under multi: $a \in F$, $u_1 \in U_1, \ldots, u_m \in U_m$ imply $au_1 \in U_1, \ldots, au_m \in U_m$

① We have $0 = \underbrace{0 + \ldots + 0}_{m} \in U_1 + \ldots U_m$

So $U_1 + \ldots + U_m$ contains the additive identity

② We also have:

If $u, w \in U_1 + \ldots + U_m$ then we can write $u = u_1 + \ldots + u_m$ and $w = w_1 + \ldots w_m$.

So $u + w = (u_1 + \ldots + u_m) + (w_1 + \ldots + w_m)$
$= (u_1 + w_1) + \ldots + (u_m + w_m)$
$\in U_1 + \ldots + U_m$.

Therefore $U_1 + \ldots + U_m$ is closed under add.

③ Let $a \in F$

$au = a(u_1 + \ldots + u_m)$
$= au_1 + \ldots + au_m$
$\in U_1 + \ldots + U_m$

therefore, $U_1 + \ldots + U_m$ is closed under scalar multi.

So $U_1 + \ldots + U_m$ a subspace of V.

Let $i = 1, \ldots, m$ let $x_i \in U_i$ then,

$$x_i = \underbrace{0 + \ldots + 0}_{i-1 \text{ terms}} + \underbrace{x_i}_{i^{th} \text{ term}} + \underbrace{0 + \ldots + 0}_{m-i \text{ terms}}$$

$\in U_1 + \ldots + U_{i-1} + U_i + U_{i+1} \ldots + U_m = U_1 + \ldots + U_m$

Observe that every subspace of V that contains $U_1, \ldots, U_m$ must also contain finite sums of elements of $U_1, \ldots, U_m$

(This is due to the property of closed under add. for subspaces.

This means in particular that every subspace that contains $U_1, \ldots, U_m$ must contain subspace $U_1 + \cdots + U_m$

Since $U_1 + \ldots + U_m$ is contained in every subspace that contains $U_1, \ldots, U_m$.

So $U_1, + + , U_m$ is the smallest subspace that contains $U_1, \ldots, U_m$.

$\boxed{1.4}$ Direct Sums

Def. Let $U_1 \ldots U_m$ be subspaces of V then,

- $U_1 + + U_m$ is a direct sum if each element of $U_1 + + U_m$ is written in only one way as the sum

$$U_1 + + U_m, \text{ where}$$

$$U_1 \in U_1 \ldots U_m \in U_m$$

- $U_1 + \ldots + U_m$ is the notation to denote the direct sum.

$\boxed{1.44}$ Condition for a direct sum.

Let $U_1 \ldots U_m$ be subspaces of V. Then $U_1 + U_m$ is a direct sum IFF the only way to write 0 as a sum $u_1 + + u_m$, where $u_1 \in U_1 \ldots u_m \in U_m$, is by taking $u_1 = 0, \ldots, u_m = 0$

Proof: Forward direction: If $U_1 + + U_m$ is a direct sum then the only way to write 0 is taking $u_1 = 0, \ldots, u_m = 0$

Suppose $U_1 + + U_m$ is a direct sum. Then, by definition of the direct sum, we can only write 0 as a sum $u_1 + + u_m$ $(0 = u_1 + + u_m)$

Backward direction: If only way to write 0 as a sum $u_1 + + u_m$ is by taking $u_1 = 0, \ldots u_m = 0$, then $U_1 + + U_m$ is a direct sum.

Suppose the only way write $0 = u_1 + \cdots + u_m$ is to trade $u_1 = 0, \ldots, u_m = 0$.

We claim that $U_1 + \cdots + U_m$ is a direct sum. Let $v \in U_1 + \cdots + U_m$.

$\star_1$ $v = u_1 + \cdots + u_m$. where $u_1 \in U_1, \ldots, u_m \in U_m$

We need to show that this representation is unique. To show this, consider another representation

$\star_2$ $v = v_1 + \cdots + v_m$ where $v_1 \in U_1, \ldots, v_m \in U_m$.

Subtract $\star_1$ and $\star_2$ we get

$$0 = (u_1 - v_1) + \cdots + (u_m - v_m)$$

Since we assumed in the backward direction that the only way to write 0 as a sum $u_1 + \cdots + u_m$ is to take $u_1 = 0, \ldots, u_m = 0$, we have from $\star_1 - \star_2$ that we need to take.

$$u_1 - v_1 = 0, \ldots, u_m - v_m = 0$$

so $u_1 = v_1, \ldots, u_m = v_m$.

so $v = v_1 + \cdots + v_m$

$= u_1 + \cdots + u_m$

Therefore, our representation of v is unique, that is, written in only one way.

So $U_1 + \cdots + U_m$ is a direct sum.

$\boxed{1.45}$ Direct sum of two subspaces.

Suppose U and W are subspaces of V. Then $U+W$ is a direct sum IFF $U \cap W = \{0\}$

Forward direction: If $U+W$ is a direct sum, then $U \cap W = \{0\}$

Suppose $U+W$ is a direct sum. Suppose $v \in U \cap W$

Then $0 = v + (-v)$

Where $v \in U$ and $-v \in W$. Since $0 \in U+W$, by the definition of direct sum, we can write 0 in only one way. Namely, we conclude $v = 0$. Therefore $U \cap W \subset \{0\}$

At the same time, we know that $U \cap W$ is a subspace of V, which means in particular $0 \in U \cap W$, or $\{0\} \subset U \cap W$.

Since $U \cap W \subset \{0\}$ and $\{0\} \subset U \cap W$, we conclude $U \cap W = \{0\}$

Backward direction: If $U \cap W = \{0\}$, then $U+W$ is direct sum.

Suppose $U \cap W = \{0\}$ let $u \in U$, $w \in W$ satisfy

$$0 = u + w.$$

But $0 = u + w$ implies

$$u = -w \in W$$

So $u \in U$ and $u \in W$; namely, $u \in U \cap W$. So $u \in \{0\}$, and so $u = 0$. And $0 = u + w$ with $u = 0$ implies $w = 0$.

So the only way to write 0 as a sum $u + w$ is to take $u = 0$, $w = 0$

By 1.44 $U+W$ is a direct sum $\blacksquare$