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1. C Subspaces

1.32 Definition: Let U be a subset of V.
Let V be a vector space (with addition and scalar multiplication)
Then U is a $\boxed{\text{subspace}}$ of V if U is a vector space with the same addition and scalar multiplication.

1.34 Conditions of subspace:
U is a subspace of V if and only if it satisfies :
Additive identity: $0 \in U$

Closed under addition: $u, w \in U$ implies $u + w \in U$.

Closed under scalar multiplication: $\alpha \in \mathbb{F}$ and $u \in U$ implies $\alpha \cdot u \in U$

1.35 Examples of subspaces

a). If $b \in \mathbb{F}$, then $U_1 = \{(x_1, x_2, x_3, x_4) \in \mathbb{F}^4 \underset{\text{s.t.}}{:} x_3 = 5x_4 + b\}$

is a subspace of $\mathbb{F}^4$ if and only if $b = 0$

Proof:

- Additive identity: $0 \in U$, $\overset{x_1\ x_2\ x_3\ x_4}{(0, 0, 0, 0)} \in U_1$ - ?

$$x_3 = 5x_4 + b$$
$$0 = 5 \cdot (0) + b$$
$$0 = b$$

So additive identity is satisfied if and only if $b = 0$.

Otherwise, if $b \neq 0$, then the statement $0 = b$ would be false.

$0 = b \neq 0$, contradiction.

- Closed under addition

Let $(x_1, x_2, x_3, x_4), (y_1, y_2, y_3, y_4) \in U_1$.

Then $x_3 = 5x_4 + b$, $y_3 = 5y_4 + b$.

So we have: $x_3 + y_3 = (5x_4 + b) + (5y_4 + b) = 5(x_4 + y_4) + 2b$

If $b = 0$, we have $x_3 = 5x_4$, $y_3 = 5y_4$

and, $x_3 + y_3 = 5(x_4 + y_4)$

So if $b = 0$, then $(x_1, x_2, x_3, x_4) + (y_1, y_2, y_3, y_4) =$
$(x_1 + y_1, x_2 + y_2, x_3 + y_3, x_4 + y_4) \in U_1$

So if $b = 0$, then U_1 is closed under addition.

- Closed under multiplication:

let $\alpha \in \mathbb{F}$ and $(x_1, x_2, x_3, x_4) \in U$

Then $x_3 = 5x_4 + b$.

$\alpha x_3 = 5\alpha x_4 + \alpha b$

If $b=0$, $\alpha x_3 = 5\alpha x_4$

So if $b=0$, then:

$a(x_1, x_2, x_3, x_4) = (ax_1, ax_2, ax_3, ax_4) \in U_1$

So U_1 is closed under scalar multiplication.

Therefore, U_1 is a subspace of V.

b). Let U_2 be the set of continuous real-valued functions. on the interval $[0,1]$. Then U_2 is a subspace of $\mathbb{R}^{[0,1]} = \{f: [0,1] \to \mathbb{R}\}$

- Additive identity:

  The zero function is continuous. So $0 \in U_2$.

- Closed under addition:

  Let $f, g \in U_2$ (Let $f, g: [0,1] \to \mathbb{R}$ be continuous)

  Sum of two continuous functions is continuous.

  So $f+g \in U_2$

- Closed under multiplication

  Let $a \in \mathbb{F}$, $f \in U_2$ (Let $f: [0,1] \to \mathbb{R}$ be continuous)

  Scalar multiplication of continuous funtions is continuous.

  So $a \cdot f \in U_2$

So, U_2 is a subspace of $\mathbb{R}^{[0,1]}$.

Sums of subspace:

1.36 Definition: sum of subsets.

Let $U_1, \cdots, U_m$ be subsets of V. The sum of $U_1, \cdots, U_m$ is denoted $U_1 + \cdots + U_m$

and is the set of all possible sums of elements of $U_1, \cdots, U_m$

In other words,

$$U_1 + \cdots + U_m = \{u_1 + \cdots + u_m : u_1 \in U_1, \cdots, u_m \in U_m\}$$

1.37 Example: Let $\mathbb{F}^3$ be a vector space.

Let $U = \{(x,0,0) \in \mathbb{F}^3 : x \in \mathbb{F}\}$

and $W = \{(0,y,0) \in \mathbb{F}^3 : y \in \mathbb{F}\}$

then $U + W = \{(x,y,0) \in \mathbb{F}^3 : x, y \in \mathbb{F}\}$

$(x,0,0) \in U$, $(0,y,0) \in W$

$(x,0,0) + (0,y,0) \in U + W \Rightarrow (x,y,0) \in U + W$

1.39 Sum of subspaces is the smallest containing subspace.

Suppose $U_1, \cdots, U_m$ are subspaces of V. Prove that $U_1 + \cdots + U_m$ is the SMALLEST subspace of V containing $U_1, \cdots, U_m$

Proof:

First, we claim that $U_1 + \cdots + U_m$ is a subspace of V. Since $U_1, \cdots, U_m$ are subspaces of V, we have:

- Additive identities: $0 \in U_1, \cdots, 0 \in U_m$.
- Closed under addition: $U_1, W_1 \in V_1$, $\cdots$, $U_m, W_m \in V_m$ imply $U_1 + W_1 \in V_1, \cdots, U_m + W_m \in V_m$
- Closed under scalar multiplication: $a \in \mathbb{F}$, $U_1 \in V_1$, $\cdots$, $U_m \in V_m$

imply $au_1 \in V_1, \cdots, au_m \in V_m$.

- We have

$$0 = \underbrace{0 + \cdots + 0}_{m} \in U_1 + \cdots + U_m$$

So $U_1 + \cdots + U_m$ contains the additive identity.

- We also have:

If $u, w \in U_1 + \cdots + U_m$ then we can write

$u = u_1 + \cdots + u_m$ and $w = w_1 + \cdots + w_m$.

So $u + w = (u_1 + \cdots + u_m) + (w_1 + \cdots + w_m)$

$= (u_1 + w_1) + \cdots + (u_m + w_m)$

$\in U_1 + \cdots + U_m$

Therefore, $U_1 + \cdots + U_m$ is closed under addition.

- Let $a \in \mathbb{F}$, we also have

$au = a(u_1 + \cdots + u_m)$

$= au_1 + \cdots + au_m \quad \in U_1 + \cdots + U_m$

Therefore, $U_1 + \cdots + U_m$ is closed under scalar multiplication

So $U_1 + \cdots + U_m$ is a subspace of V.

- Next, we claim that $U_1 + \cdots + U_m$ is the smallest subspace of V containing $U_1, \cdots, U_m$

Let $i = 1, \cdots, m$ let $x_i \in U_i$ then:

$$x_i = \underbrace{0 + \cdots + 0}_{i-1} + \underbrace{x_i}_{i^{th}} + \underbrace{0 \cdots + 0}_{m-i} \in U_1 + \cdots + U_{i-1} + U_i + \cdots + U_m$$

$$= U_1 + \cdots + U_m$$

Observe that every subspace of V that contains $U_1, \cdots, U_m$ must also contain finite sums of elements of $U_1, \cdots, U_m$

(This is due to the property of closed under addition for subspaces)

This means in particular that every subspace that contains $U_1, \cdots, U_m$ must contain the subspace $U_1 + \cdots + U_m$

Since $U_1 + \cdots + U_m$ is contained in every subspace that contains $U_1, \cdots, U_m$, So $U_1 + \cdots + U_m$ is the smallest subspace that contains $U_1, \cdots, U_m$.

Direct Sums:

1.40 Definition:

Let $U_1, \cdots, U_m$ be subspaces of V. Then $U_1 + \cdots + U_m$ is a

- direct sum if each element of $U_1 + \cdots + U_m$ is written in only one way as the sum $u_1 + \cdots + u_m$, where $u_1 \in V_1, \cdots, u_m \in V_m$
- $U_1 \oplus \cdots \oplus U_m$ is the notation to denote the direct sum.

1.44 Condition for a direct sum.

Let $U_1, \cdots, U_m$ be subspaces of V. Then $V_1 + \cdots + V_m$ is a direct sum if and only if the only way to write 0 as a sum $u_1 + \cdots + u_m$, where $u_1 \in V_1, \cdots, u_m \in V_m$, is by taking $u_1 = 0, \cdots, u_m = 0$

For ward direction:

If $u_1 + \cdots + u_m$ is a direct sum, then the only way to write 0 is taking $u_1 = 0, \cdots, u_n = 0$

Suppose $U_1 \oplus \cdots \oplus U_m$ is a direct sum. Then by definition of the direct sum, we can only write 0 as a sum $U_1 + \cdots + U_m$ $(0 = U_1 + \cdots + U_m)$ in one way: by taking $u_1 = 0, \cdots, u_m = 0$

Backward direction:

If the only way to write 0 as a sum $u_1+\cdots+u_m$ is by taking $u_1=0,\cdots,u_m=0$, then $U_1+\cdots+U_m$ is a direct sum.

Suppose the only way to write $0=u_1+\cdots+u_m$ is to take $u_1=0,\cdots,u_m=0$.

We claim that $U_1+\cdots+U_m$ is a direct sum. Let $v\in U_1+\cdots+U_m$

Then by 1.40, we can write.

☆ $V=u_1+\cdots+u_m$, $u_1\in V_1,\cdots,u_m\in V_m$

We need to show that this representation is unique.

To show this, consider another representation

☆ $V=v_1+\cdots+v_m$ where $v_1\in V_1,\cdots,v_m\in V_m$

Substract ☆ & ☆, we get

$0=(u_1-v_1)+\cdots+(u_m-v_m)$

Since we assumed in the backward direction that the only way to write 0 as a sum $u_1+\cdots+u_m$ is to take $u_1=0,\cdots,u_m=0$, we have from ☆ − ☆ that we need to take $u_1-v_1=0,\cdots,u_m-v_m=0$

So $u_1=v_1,\cdots,u_m=v_m$

So $V=v_1+\cdots v_m=u_1+\cdots+u_m$

Therefore, our representation of v is unique, that is, written in only one way.

So $U_1+\cdots+U_m$ is a direct sum.

1.45 Direct sum of two subspaces.

Suppose U and W are subspaces of V. Then $U+W$ is a direct sum if and only if $U\cap W=\{0\}$

- Forward direction: if $U+W$ is a direct sum, then $U\cap W=\{0\}$.

Suppose $U+W$ is a direct sum. Suppose $v\in U\cap W$

Then $0=v+(-v)$

where $v\in U$ and $-v\in W$, since $0\in U+W$. by the definition of direct sum, we can write 0 in only one way. Namely we conclude $v=0$. Therefore $U\cap W\subset\{0\}$

At the same time, we know that $U\cap W$ is a subspace of V, which means in particular $0\in U\cap W$, or $\{0\}\subset U\cap W$.

Since $U\cap W\subset\{0\}$ and $\{0\}\subset U\cap W$, we conclude $U\cap W=\{0\}$.

- Backward direction: If $U\cap W=\{0\}$, then $U+W$ is a direct sum.

Suppose $U\cap W=\{0\}$. Let $u\in V$, $w\in W$. satisfy.

$0=u+w$

- But $0=u+w$ implies $u=w\in W$.

So $u\in V$ and $u\in W$, namely, $u\in U\cap W$, so $u\in\{0\}$ and so $u=0$.

And $0=u+w$ with $u=0$, implies $w=0$, so the only way to write 0 as a sum $u+w$ is to take $u=0$, $w=0$

By 1.44, $U+W$ is a direct sum.

Discussion.

set containment proofs.

Ex1. Prove $(A\setminus B)\cup(C\setminus B)=(A\cup C)\setminus B$