1C Subspaces

1.32 Def: - Let V be a vector space (with addition & scalar multiplication).
- Let U be a subset of V. Then U is a $\boxed{\text{subspace}}$ of V if U is also a vector space with the same addition and scalar multiplication.

1.34 Conditions of subspace:

U is a subspace of V if and only if it satisfies:

ADDITIVE IDENTITY: $0 \in U$

CLOSED UNDER ADDITION: $u, w \in U$ implies $u + w \in U$

CLOSED UNDER SCALAR MULTIPLICATION:

$a \in \mathbb{F}$ and $u \in U$ implies $au \in U$.

$(\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C})$

Eg 1.35 Examples of Subspaces.

(a) If $b \in \mathbb{F}$, then $U_1 = \{(x_1, x_2, x_3, x_4) \in \mathbb{F}^4 :$ (in) (such that)

$x_3 = 5x_4 + b\}$

is a subspace of $\mathbb{F}^4$ iff $b = 0$.

proof: Additive Identity $0 \in U$

$(\overset{x_1}{0}, \overset{x_2}{0}, \overset{x_3}{0}, \overset{x_4}{0}) \in U_1$?

$x_3 = 5x_4 + b$

$0 = 5(0) + b$

$0 = b$

So additive identity is satisfies iff $b = 0$

(otherwise, if $b \neq 0$, then the statement $0 = b$ would be false.

$0 = b \neq 0$

contradiction)

(b) Closed under addition

Let $(x_1, x_2, x_3, x_4), (y_1, y_2, y_3, y_4) \in U_1$.

Then $x_3 = 5x_4 + b$ and $y_3 = 5y_4 + b$.

So we have:

$$x_3 + y_3 = (5x_4 + b) + (5y_4 + b)$$
$$= 5(x_4 + y_4) + 2b$$

If $b = 0$, we have $x_3 = 5x_4$ and $y_3 = 5y_4$, and

$$x_3 + y_3 = 5(x_4 + y_4)$$

So if $b = 0$, then

$$(x_1, x_2, x_3, x_4) + (y_1, y_2, y_3, y_4) =$$
$$= (x_1 + y_1, x_2 + y_2, x_3 + y_3, x_4 + y_4) \in U_1$$

So if $b = 0$, then U_1 is closed under addition.

(c) Closed under scalar multiplication

Let $a \in \mathbb{F}$ and $(x_1, x_2, x_3, x_4) \in U_1$

then $a \cdot (x_3) = (5x_4 + b) \cdot a$

so we have $ax_3 = 5ax_4 + ab$

If $b = 0$, then $x_3 = 5x_4$ and $ax_3 = 5ax_4$

So if $b = 0$ then

$$a(x_1, x_2, x_3, x_4) = (ax_1, ax_2, ax_3, ax_4)$$

So U_1 is closed under scalar multiplication.

Therefore U_1 is a subspace of V.

Eg2. Let U_2 be the set of continuous real-valued functions on the interval $[0, 1]$. Then U_2 is a subspace of $\mathbb{R}^{[0,1]} = \{ f : [0,1] \to \mathbb{R} \}$

- Additive Identity

  The zero function is continuous. So $0 \in U_2$

- Closed under Addition

  Let $f, g \in U_2$ (let $f, g : [0,1] \to \mathbb{R}$ be continuous)

  so $f + g \in U_2$

Closed under Scalar multiplication

Let $a \in \mathbb{F}$, $f \in U_2$. (let $f : [0,1] \to \mathbb{R}$ be continuous.)

Scalar multiples of continuous functions is continuous

So $af \in U_2$

So U_2 is a subspace of $\mathbb{R}^{[0,1]}$

Sums of Subspace

1.36 Def: Sum of subsets

Let $U_1, \ldots, U_m$ be subsets of V. The sum of $U_1, \ldots, U_m$ is denoted

$$U_1 + \cdots + U_m$$

and is the set of all possible sums of elements of $U_1, \ldots, U_m$

$\Rightarrow$ In other words,

$$U_1 + \cdots + U_m = \{u_1 + \cdots u_m : u_1 \in U_1, \ldots, u_m \in U_m\}$$

1.37 Eg Let $\mathbb{F}^3$ be a vector space.

Let $U = \{(x,0,0) \in \mathbb{F}^3 : x \in \mathbb{F}\}$

and $W = \{(0,y,0) \in \mathbb{F}^3 : y \in \mathbb{F}\}$

then $U + W = \{(x,y,0) : x, y \in \mathbb{F}\}$

$(x,0,0) \in U$

$(0,y,0) \in W$. ✓

$\underbrace{(x,0,0)}_{\in U} + \underbrace{(0,y,0)}_{\in W} \in U + W$

$= (x+0, 0+y, 0+0) \in U + W$

$= (x,y,0) \in U+W$

Subspace

* 1.39 Eg Sum of subspaces is the smallest containing subspace

Suppose $U_1, \dots, U_m$ are subspace of V. Prove that $U_1 + \dots + U_m$ is the SMALLEST subspace of V contains $U_1, \dots, U_m$

Proof: First[1] we claim that $U_1 + \dots + U_m$ is a subspace of V.

Since $U_1, \dots, U_m$ are subspaces of V, we have:

ADDITIVE IDENTITIES: $0 \in U_1, \dots, 0 \in U_m$.

CLOSED UNDER ADDITION: $u_1 w_1 \in U_1, \dots u_m w_m \in U_m$ imply $u_1 + w_1 \in U_1, \dots, u_m + w_m \in U_m$

CLOSED UNDER SCALAR MULTIPLICATION

$a \in \mathbb{F}$, $u_1 \in U_1, \dots, u_m \in U_m$ imply $au_1 \in U_1, \dots, au_m \in U_m$

$\Rightarrow$ we have:

$$0 = \underbrace{0 + \dots + 0}_{m} \in U_1 + \dots + U_m$$

So $U_1 + \dots + U_m$ contains the additive identity

$\Rightarrow$ we also have:

If $u, w \in U_1 + \dots + U_m$ then we can write

$u = u_1 + \dots + u_m$ and $w = w_1 + \dots + w_m$

So $u + w = (u_1 + \dots + u_m) + (w_1 + \dots + w_m)$

$= (u_1 + w_1) + \dots + (u_m + w_m) \in U_1 + \dots U_m$

Therefore $U_1 + \dots + U_m$ is closed under addition.

Let $a \in \mathbb{F}$

$\Rightarrow$ we also have

$au = a(u_1 + \dots u_m)$

$= au_1 + \dots au_m \in U_1 + \dots U_m$

Therefore, $U_1 + \dots U_m$ is close under scalar multiplication

So $U_1 + \dots + U_m$ is a subspace of V.

next ② we claim that $U_1 + \cdots + U_m$ is the SMALLEST subspace of V containing $U_1, \ldots, U_m$.

Let $i = 1, \ldots, m$. Let $x_i \in U_i$. Then

$$x_i = \underbrace{0 + \cdots + 0}_{i-1 \text{ term}} + \underbrace{x_i}_{i^{th} \text{ term}} + \underbrace{0 + \cdots + 0}_{m-i \text{ term}}$$

$$\in U_1 + \cdots + U_{i-1} + U_i + U_{i+1} + \cdots U_m$$

$$:= U_1 + \cdots + U_m$$

Observe that every subspace of V that contains $U_1, \ldots, U_m$ must also contain finite sums of elements of $U_1, \ldots, U_m$.

(This is due to the property of closed under addition for subspaces.)

⇒ This means in particular that every subspace that contains $U_1, \ldots, U_m$ must contain the subspace $U_1 + \cdots + U_m$.

Since $U_1 + \cdots + U_m$ is contained in every subspace that contains $U_1, \ldots, U_m$.

So $U_1 + \cdots + U_m$ is ~~contained~~ the SMALLEST subspace that contains $U_1, \ldots, U_m$.

Direct Sums

1.40 Def direct sum

⇒ Let $U_1, \ldots, U_m$ be subspaces of V. Then

- $U_1 + \cdots + U_m$ is a direct sum is each element of $U_1 + \cdots + U_m$ is written in ONLY ONE WAY as the sum
  $$u_1 + \cdots + u_m,$$
  where $u_1 \in U_1, \ldots, u_m \in U_m$.
- $U_1 \oplus \cdots \oplus U_m$ is the NOTATION to denote the direct sum

1.44 Condition for a direct Sum

Let $U_1, \dots, U_m$ be subspaces of V. Then $U_1 + \dots + U_m$ is a direct sum if and only if the only way to write 0 as a sum $u_1 + \dots + u_m$, where $u_1 \in U_1, \dots, u_m \in U_m$, is by taking $u_1 = 0, \dots, u_m = 0$.

Proof: (a) Forward Direction:

If $U_1 + \dots + U_m$ is a direct sum, then the only way to write 0 as a sum $u_1 + \dots + u_m$ is taking $u_1 = 0, \dots, u_m = 0$

Suppose $U_1 \oplus \dots \oplus U_m$ is a direct sum. Then, by definition the direct sum, we can only write 0 as a sum $u_1 + \dots + u_m$ ($0 = u_1 + \dots + u_m$) in one way: by taking $u_1 = 0, \dots, u_m = 0$

(b) Backward Direction:

If the only way to write 0 as a sum $u_1 + \dots + u_m$ is by taking $u_1 = 0, \dots, u_m = 0$, then $U_1 + \dots + U_m$ is a direct sum.

Suppose that the only way to write $0 = u_1 + \dots + u_m$ is to $u_1 = 0, \dots, u_m = 0$.

We claim that $U_1 + \dots + U_m$ is a direct sum.

Let $v \in U_1 + \dots + U_m$.

Then by 1.40, we can write

* $v = u_1 + \dots + u_m$.

(where $u_1 \in U_1, \dots, u_m \in U_m$)

we need to show this representation is unique. To show this, consider ANOTHER representation

☆ $v = v_1 + \cdots v_m$ where $v_1 \in U_1, \cdots, v_m \in U_m$

Subtract ☆ and ☆, we have

$$0 = (u_1 - v_1) + \cdots + (u_m - v_m)$$

Since we assumed in the ~~background~~ backward direction that the only way to write 0 as a sum $u_1 + \cdots + u_m$ is to take $u_1 = 0, \cdots, u_m = 0$, we have from ☆ - ☆ that we need to take

$$u_1 - v_1 = 0, \ldots u_m - v_m = 0.$$

so $u_1 = v_1, \cdots, u_m = v_m$.

so $v = v_1 + \cdots + v_m$
$= u_1 + \cdots + u_m$

Therefore, our representation of v is unique; that is, written in ONLY one way.

so $u_1 + \cdots + u_m$ is a direct sum.

1.45 Direct Sum of two subspaces

Suppose U and W are subspaces of V. Then U+W is a direct sum if and only if $U \cap W = \{0\}$.

Forward Direction: If U+W is a direct sum, then $U \cap W = \{0\}$

Suppose U+W is a direct sum. Suppose $v \in U \cap W$.

Then $0 = v + (-v)$,

where $v \in U$ and $-v \in W$. Since $0 \in U + W$, by the definition of direct sum, we can write 0 in ONLY ONE WAY.

Namely, we conclude $v=0$. Therefore $U \cap W \subset \{0\}$

At the same time, we know that $U \cap W$ is a subspace of V, which means in particular $0 \in U \cap W$, or $\{0\} \subset U \cap W$.

Since $U \cap W \subset \{0\}$ and $\{0\} \subset U \cap W$, we conclude $U \cap W = \{0\}$.

Backward Direction: If $U \cap W = \{0\}$, then $U+W$ is a direct sum.

Suppose $U \cap W = \{0\}$. Let $u \in U$, $w \in W$ satisfy

$$0 = u+w.$$

But $0 = u+w$ implies

$$u = -w \in W$$

So $u \in U$ and $u \in W$; namely, $u \in U \cap W$.

So $u \in \{0\}$, and so $u=0$.

And $0 = u+w$ with $u=0$ implies $w=0$.

So the only way to write 0 as a sum $u+w$ is to take $u=0$, $w=0$,

By 1.44, $U+W$ is a direct sum