IV) $\alpha\vec{0} = \vec{0} \quad \forall \alpha \in F$

proof:
$\cancel{\alpha\vec{0}} = \alpha(\vec{0}+\vec{0}) = \cancel{\alpha\vec{0}} + \alpha\vec{0} \quad \Rightarrow \boxed{\alpha\vec{0} = \vec{0}}$

V) the # (-1) times a vector
$(-1)\vec{v} = -\vec{v}$ for any $\vec{v} \in V$

$\vec{v}+\vec{w} = 0$ unique
$w = -v$

$v + (-v) = 0$
$\vec{v} + (-1)\vec{v} = 0$?

use $\alpha\vec{v} + \beta\vec{v} = (\alpha+\beta)\vec{v}$
$(1+(-1))\vec{v}$
$= 0\vec{v} = \vec{0}$ ✓

Ex 1.B

2) suppose $a \in F$, $v \in V$ & $a\vec{v} = \vec{0}$. prove that $a=0$ or $\vec{v}=\vec{0}$

If $a=0$ ✓
what if $a \neq 0 \overset{?}{\Rightarrow} \vec{v} = \vec{0}$
$a \neq 0$ so $\frac{1}{a}$ is well defined

$\frac{1}{a}(a\vec{v}) = (\frac{1}{a}\vec{0})$
$(\frac{1}{a}a)\vec{v} = \vec{0}$
$\boxed{\vec{v} = \vec{0}}$ ✓

6/26/19
week1
wednesday

Email: ryanta@math.ucr.edu
OH: MTWR 11:10 AM - 12:00PM at CHASS 2134
Quiz 1 in discussion today
Direct Proofs
3:15pm - 4:00pm
open book, open notes

Group exam 1 tomorrow
- 20 pts on definitions
- 40 pts on 2 Axler exercises
  - look at: Axler exercises 1.C.1 & 1.C.24
- 40 pts on curveballs
  - know & apply properties of vector space & subspace

1.C Subspaces

let V be a vector space (w/ addition & scalar multiplicati

1.32 Definition: let U be a subset of V. Then U is a $\boxed{\text{subspace}}$ of V if U is also a vector space w/ the same addition & scalar multiplication

1.34 Conditions of subspace
U is a subspace of V if & only if it satisfies:
Additive Identity: $0 \in U$
closed under addition: $u, w \in U$ implies $u+w \in U$
scalar multiplication: $a \in F$ & $u \in U$ implies $au \in U$

1.35 Examples of subspaces

a) if $b \in \mathbb{F}$, then $U_1 = \{(x_1, x_2, x_3, x_4) \in \mathbb{F}^4 : x_3 = 5x_4 + b\}$
is a subspace of $\mathbb{F}^4$ if & only if $b = 0$

in such that

<u>Proof</u>. Additive Identity $0 \in U$

$$\overset{x_1}{(0}, \overset{x_2}{0}, \overset{x_3}{0}, \overset{x_4}{0}) \in U_1$$

$x_3 = 5x_4 + b$ Additive identity is satisfied if & only if $b = 0$

$0 = 5(0) + b$

$0 = b$. (otherwise, if $b \neq 0$, then the statement $0 = b$ would false. $0 = b \neq 0$
∴ contradiction

<u>closed under addition</u>

let $(x_1, x_2, x_3, x_4), (y_1, y_2, y_3, y_4) \in U_1$
Then $x_3 = 5x_4 + b$ & $y_3 = 5y_4 + b$
so we have

$$x_3 + y_3 = (5x_4 + b) + (5y_4 + b)$$
$$= 5(x_4 + y_4) + 2b$$

If $b = 0$, we have $x_3 = 5x_4$ & $y_3 = 5y_4$ & $x_3 + y_3 = 5(x_4 + y_4)$
so if $b = 0$, then

$$(x_1, x_2, x_3, x_4) + (y_1, y_2, y_3, y_4) = (x_1 + y_1, x_2 + y_2, x_3 + y_3, x_4 + y_4) \in U_1$$

so if $b = 0$, then U_1 is closed under addition

<u>closed under scalar multiplication</u>

let $a \in \mathbb{F}$ & $(x_1, x_2, x_3, x_4) \in U_1$
Then $x_3 = 5x_4 + b$
so we have $a(x_3 = 5x_4 + b)$

$$ax_3 = 5ax_4 + ab$$

If $b = 0$, then $x_3 = 5x_4$ & $ax_3 = 5(ax_4)$
so if $b = 0$, then

$$a(x_1, x_2, x_3, x_4) = (ax_1, ax_2, ax_3, ax_4) \in U_1$$

so U_1 is closed under scalar multiplication
Therefore, U_1 is a subspace of V.

b) Let U_2 be the set of continuous real-valued functions on the interval $[0,1]$. Then U_2 is a subspace of $\mathbb{R}^{[0,1]} = \{f:[0,1] \to \mathbb{R}$

Additive Identity

The zero function is continuous so $0 \in U_2$

closed under addition

Let $f, g \in U_2$ (Let $f, g: [0,1] \to \mathbb{R}$ be continuous)

sum of two continuous functions is continuous

so $f+g \in U_2$

closed under scalar multi.

Let $a \in \mathbb{F}$, $f \in U_2$ (Let $f: [0,1] \to \mathbb{R}$ be continuous)

scalar multiple of cont. functions is cont.

so $af \in U_2$

so U_2 is a subspace of $\mathbb{R}^{[0,1]}$

Sums of subspaces

1.36 Definition: sum of subsets

Let $U_1, \ldots, U_m$ be subsets of V. The sum of $U_1, \ldots, U_m$ is denoted $U_1 + \ldots + U_m$

& is the set of all possible sums of elements of $U_1, \ldots, U_m$

In other words,

$$U_1 + \ldots + U_m = \{u_1 + \ldots + u_m : u_1 \in U_1, \ldots, u_m \in U_m\}$$

1.37 Example

Let $\mathbb{F}^3$ be a vector space

Let $V = \{(x,0,0) \in \mathbb{F}^3 : x \in \mathbb{F}\}$

& $W = \{(0,y,0) \in \mathbb{F}^3 : y \in \mathbb{F}\}$

Then $U + W = \{(x,y,0) : x, y \in \mathbb{F}\}$

$\underbrace{(x,0,0)}_{\in U} + \underbrace{(0,y,0)}_{\in W} \in U + W$ ✓

$= (x+0, 0+y, 0+0) \in U+$

$= (x,y,0) \in U + W$

$(x,0,0) \in U$

$(0,y,0) \in W$

1.39 sum of subspaces is the smallest containing subspace

Suppose $U_1, \ldots, U_m$ are subspaces of V. Prove that $U_1 + \ldots + U_m$ is the SMALLEST subspace of V contain $U_1, \ldots, U_m$

Proof: First, we claim that $U_1 + \ldots + U_m$ is a subspace of V

since $U_1, \ldots, U_m$ are subspaces of V, we have:

Additive Identities: $0 \in U_1, \ldots, 0 \in U_m$

Closed under Addition: $u_1, w_1 \in U_1, \ldots, u_m, w_m \in U_m$

imply $u_1 + w_1 \in U_1, \ldots, u_m + w_m \in U_m$

closed under scalar mult.

$a \in F, u_1 \in U_1, \dots, u_m \in U_m$

imply $au_1 \in U_1, \dots, au_m \in U_m$

we have

$0 = \underbrace{0 + \dots + 0}_{m} \in U_1 + \dots + U_m$

so $U_1 + \dots + U_m$ contains the additive identity

we also have:

If $u, w \in U_1 + \dots + U_m$ then we can write

$u = u_1 + \dots + u_m$ & $w = w_1 + \dots + w_m$

so

$u + w = (u_1 + \dots + u_m) + (w_1 + \dots + w_m)$

$= (u_1 + w_1) + \dots + (u_m + w_m)$

$\in U_1 + \dots + \in U_m$

Therefore, $U_1 + \dots + U_m$ is closed under addition

Let $a \in F$ we also have

$au = a(u_1 + \dots + u_m)$

$= au_1 + \dots + au_m$

$\in U_1 + \dots + U_m$

Therefore, $U_1 + \dots + U_m$ is closed under scalar mult.

so $U_1 + \dots + U_m$ a subspace of V.

Next we claim that $U_1 + \dots + U_m$ is the SMALLEST subspace of V containing $U_1, \dots, U_m$

Let $i = 1, \dots, m$ Let $x_i \in U_i$ then

$x_i = \underbrace{0 + \dots + 0}_{i-1 \text{ terms}} + \underbrace{x_i}_{i\text{th term}} + \underbrace{0 + \dots + 0}_{m-i \text{ terms}}$

$\in U_1 + \dots + U_{i-1} + U_i + U_{i+1} + \dots + U_m$

$= U_1 + \dots + U_m$

observe that every subspace of V that contains $U_1, \dots, U_m$ must also contain finite sums elements of $U_1, \dots, U_m$ (This is due to the property of closed under addition for subspaces)

This means in particular that every subspace that contains $U_1, \dots, U_m$ must contain the subspace $U_1 + \dots + U_m$

Since $U_1 + \dots + U_m$ is contained in every subspace that contains $U_1, \dots, U_m$ so $U_1 + \dots + U_m$ is the SMALLEST subspace that contains $U_1, \dots, U_m$

Direct Sums

1.40 Def: direct sum

Let $U_1, \ldots, U_m$ be subspaces of V. Then

- $U_1 + \ldots + U_m$ is a direct sum if each element of $U_1 + \ldots + U_m$ is written is ONLY ONE WAY as the sum $u_1 + \ldots + u_m$,
  where $u_1 \in U_1, \ldots, u_m \in U_m$
- $U_1 \oplus \ldots \oplus U_m$ is the NOTATION to denote the direct sum

1.44 condition for a direct sum

Let $U_1, \ldots, U_m$ be subspaces of V. Then $U_1 + \ldots + U_m$ is a direct sum if & only if the only way to write 0 as a sum $u_1 + \ldots + u_m$, where $u_1 \in U_1, \ldots, u_m \in U_m$, is by taking $u_1 = 0, \ldots, u_m$

Proof:

Forward Direction: If $U_1 + \ldots + U_m$ is a direct sum, then the only way to write $\emptyset$ as a sum $u_1 + \ldots + u_m$ is taking $u_1 = 0, \ldots, u_m$

Suppose $U_1 \oplus \ldots \oplus U_m$ is a direct sum. Then, by definition of the direct sum, we can only write 0 as a sum $u_1 + \ldots + u_m$ in one way: by taking $u_1 = 0, \ldots, u_m = 0$, $(0 = u_1 + \ldots + u_m)$

Backward Dir: If the only way to write $\emptyset$ as a sum $u_1 + \ldots + u_m$ is by taking $u_1 = 0, \ldots, u_m = 0$, then $U_1 + \ldots + U_m$, then $U_1 + \ldots + U_m$ is a direct sum

Suppose the only way to write $0 = u_1 + \ldots + u_m$ is to take $u_1 = 0, \ldots, u_m = 0$ we claim that $U_1 + \ldots + U_m$ is a direct sum

Let $v \in U_1 + \ldots + U_m$

Then by 1.40, we can write

$v = u_1 + \ldots + u_m$ where $u_1 \in U_1, \ldots, u_m \in U_m$

We need to show that this representation is unique. To show this, consider ANOTHER representation

$*$ $v = v_1 + \ldots + v_m$ where $v_1 \in U_1, \ldots, v_m \in U_m$

since we assumed in the backward direction that the only way to write 0 as a sum $u_1 + \dots + u_m$ is to take $u_1 = 0, \dots, u_m = 0$, we have from * - * that we need to take

$u_1 - v_1 = 0, \dots, u_m - v_m = 0$

so $u_1 = v_1, \dots, u_m = v_m$

so $v = v_1 + \dots + v_m$

$= u_1 + \dots + u_m$

Therefore, our representation of v is unique; that is, written in ONLY one way

so $u_1 + \dots + u_m$ is a direct sum

1.45 Direct sum of two subspaces

Suppose V & W are subspaces of V. Then $U + W$ is a direct sum if & only if $U \cap W = \{0\}$

Forward direction: If $U + W$ is a direct sum, then $U \cap W = \{0\}$

Suppose $U + W$ is a direct sum. Suppose $v \in U \cap W$

Then $0 = v + (-v)$,

where $v \in U$ & $-v \in W$. Since $0 \in U + W$, by the def. of direct sum, we can write 0 in ONLY ONE WAY

Namely, we conclude $v = 0$. Therefore $U \cap W \in [0]$.

At the same time, we know that $U \cap W$ is a subspace of V, which means in particular $0 \in U \cap W$, or $\{0\} \subset U \cap W$

Since $U \cap W \subset \{0\}$ & $\{0\} \subset U \cap W$, we conclude $U \cap W = \{0\}$

Backward definition: If $U \cap W = \{0\}$, then $U + W$ is a direct sum

Suppose $U \cap W = \{0\}$. Let $u \in U$, $w \in W$ satisfy

$0 = u + w$

But $0 = u + w$ implies

$u = w \in W$

so $u \in U$ & $u \in W$; namely, $u \in U \cap W$, so $u \in \{0\}$ & so $u = 0$.

& $0 = u + w$ w/ $u = 0$ implies $w = 0$ so the only way to write 0 as a sum $u + w$ is to take $u = 0, w = 0$

By 1.44, $U + W$ is a direct sum