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LECTURE 03
06-26-19

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Office Hours : MTWR 11:10 AM - 12:00 PM @ CHASS 2134

Quiz 1 in discussion today - Direct proofs 3:15 PM - 4:00 PM
Open book, open notes

Group Exam 1 tomorrow
- 20 pts on definitions
- 40 pts on 2 Axler exercises
  ↳ Look at : Axler exercises 1.C.1 and 1.C.2
- 40 pts on curveballs
  - know and apply properties of vector space and subspace

SECTION 1.C Subspaces

1.32 Definition: Let U be a subset of V. Then U is a subspace of V if U is also a vector space with the same addition and scalar multiplication.
Let V be a vector space (with addition & scalar multiplication).

1.34 Condition of subspace
U is a subspace of V if and only if it satisfies:
ADDITIVE IDENTITY : $0 \in U$
CLOSED UNDER ADDITION : $u, w \in U$ implies $u + w \in U$
CLOSED UNDER SCALAR MULTIPLICATION : $a \in \mathbb{F}$ and $u \in U$ $au \in U$
($\mathbb{F} : \mathbb{R}$ or $\mathbb{F} : \mathbb{C}$)

1.35 Examples of subspaces
(a) If $b \in \mathbb{F}$, then

$$U_1 = \{(x_1, x_2, x_3, x_4) \in \mathbb{F}^4 : x_3 = 5x_4 + b\}$$

in such that

is a subspace of $\mathbb{F}^4$ if and only if $b = 0$
Proof Additive Identity $0 \in U$

(x_1, x_2, x_3, x_4)

$(0, 0, 0, 0) \in U_1$?

$x_3 = 5x_4 + b$

$0 = 5(0) + b$

$0 = b$

So additive identity is satisfied if and only if $b = 0$. (Otherwise, if $b \neq 0$, then the statement $0 = b$ would be false. $0 = b \neq 0$ contradiction

Closed under addition

Let $(x_1, x_2, x_3, x_4), (y_1, y_2, y_3, y_4) \in U_1$

Then $x_3 = 5x_4 + b$ and $y_3 = 5y_4 + b$

So we have

$$x_3 + y_3 = (5x_4 + b) + (5y_4 + b)$$
$$= 5(x_4 + y_4) + 2b$$

If $b = 0$, we have $x_3 = 5x_4$ and $y_3 = 5y_4$, and

$$x_3 + y_3 = 5(x_4 + y_4)$$

So if $b = 0$, then

$$(x_1, x_2, x_3, x_4) + (y_1, y_2, y_3, y_4)$$
$$= (x_1 + y_1, x_2 + y_2, x_3 + y_3, x_4 + y_4) \in U_1$$

So if $b = 0$, then U_1 is closed under addition

Closed under multiplication

Let $a \in \mathbb{F}$ and $(x_1, x_2, x_3, x_4) \in U_1$

Then $a(x_3 = 5x_4 + b)a$

So we have $ax_3 = 5ax_4 + ab$

If $b = 0$ then $x_3 = 5x_4$ and

$$ax_3 = 5(ax_4)$$

So if $b = 0$, then

$$a(x_1, x_2, x_3, x_4) = (ax_1, ax_2, ax_3, ax_4) \in U_1$$

So U_1 is closed under scalar multiplication

* Therefore U_1 is a subspace of V

(b) Let U_2 be the set of continuous real-valued functions on the interval $[0,1]$. Then U_2 is a subspace of $\mathbb{R}^{[0,1]} = \{f:[0,1] \to \mathbb{R}\}$

Additive identity

The zero function is continuous. So $0 \in U_2$

Closed under addition

Let $f, g \in U_2$ (Let $f, g : [0,1] \to \mathbb{R}$ be continuous.)
Sum of two continuous functions is continuous.
So $f+g \in U_2$

Closed under scalar multiplication

Let $a \in U_2$. (Let $f: [0,1] \to \mathbb{R}$ be continuous)
Scalar multiple of continuous functions is continuous. So $af \in U_2$

* So U_2 is a subspace of $\mathbb{R}^{[0,1]}$

Sum of subspace

1.36 Definition: sum of subsets

Let $U_1, \dots, U_m$ be subsets of V. The sum of $U_1, \dots, U_m$ is denoted

$$U_1 + \dots + U_m$$

and is the set of all possible sums of elements of $U_1, \dots, U_m$

In other words,

$$U_1 + \dots + U_m = \{u_1 + \dots + u_m : u_1 \in U_1, \dots, u_m \in U_m\}$$

1.37 Example Let $\#3$ be a vector space

Let $U = \{(x,0,0) \in \#^3 : x \in \mathbb{F}\}$
and $W = \{(0,y0) \in \#^3 : y \in \mathbb{F}\}$
then $U+W = \{(x,y,0) : x, y \in \mathbb{F}\}$

$(x,0,0) \in U$
$(0,y,0) \in W$

$\underset{\in U}{(x,0,0)} + \underset{\in W}{(0,y,0)} \in U+W$
$= (x+0, 0+y, 0+0) \in U+W$
$= (x,y,0) \in U+W$

1.39 Sum of subspaces is the smallest containing subspace

Suppose $U_1, \dots, U_m$ are subspaces of V. Prove that $U_1 + \dots + U_m$ is the smallest subspace of V contains $U_1, \dots, U_m$

Proof: First, we claim that $U_1 + \dots + U_m$ is a subspace of V

Since $U_1, \dots, U_m$ are subspaces of V, we have:

ADDITIVE IDENTITIES: $0 \in U_1, \dots, 0 \in U_m$

CLOSED UNDER ADDITION: $u_1 w_1 \in U_1, \dots, u_m w_m \in U_m$ imply $u_1 + w_1 \in U_1, \dots, u_m + w_m \in U_m$

CLOSED UNDER MULTIPLICATION: $a \in \mathbb{F}$, $u_1 \in U_1, \dots, u_m \in U_m$ imply $au_1 \in U_1, \dots, au_m \in U_m$

we have

$0 = \underbrace{0 + \dots + 0}_{m} \in U_1 + \dots + U_m$

So $U_1 + \dots + U_m$ contains the additive identity

we also have:

If $u, w \in U_1 + \dots + U_m$ then we can write

$u = u_1 + \dots + u_m$ and ~~[illegible]~~ $w = w_1 + \dots + w_m$

So $u + w = (u_1 + \dots + u_m) + (w_1 + \dots + w_m)$

$= (u_1 + w_1) + \dots + (u_m + w_m)$

$\in U_1 + \dots + U_m$

Therefore, $U_1 + \dots + U_m$ is closed under addition

Let $a \in \mathbb{F}$. We ~~[illegible]~~ also have

$au = a(u_1 + \dots + u_m)$

$= au_1 + \dots + au_m \quad \in U_1 + \dots + U_m$

Therefore, $U_1 + \dots + U_m$ is closed under scalar multiplication

* So $U_1 + \dots + U_m$ is a subspace of V

~~[illegible]~~ Next we claim that $U_1 + \dots + U_m$ is the smallest subspace of V containing $U_1, \dots, U_m$

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Let $i = 1, \dots, m$. Let $x_i \in U_i$. Then

$$x_i = \underbrace{0 + \dots + 0}_{i-1 \text{ terms}} + \underbrace{x_i}_{i^{th} \text{ term}} + \underbrace{0 + \dots + 0}_{m-i \text{ terms}}$$

$$\in U_1 + \dots + U_{i-1} + U_i + U_{i+1} + \dots + U_m$$

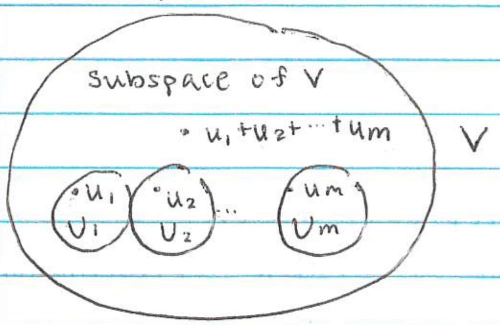
$$= U_1 + \dots + U_m$$

Observe that every subspace of V that contains $U_1, \dots, U_m$ must also contain finite sums of elements of $U_1, \dots, U_m$ (This is due to the property of closed under addition for subspaces.)

This means in particular that every subspace that contains $U_1, \dots, U_m$ must contain the subspace $U_1 + \dots + U_m$.

Since $U_1 + \dots + U_m$ is contained in every subspace that contains $U_1, \dots, U_m$.

So $U_1 + \dots + U_m$ is the SMALLEST subspace that contains $U_1, \dots, U_m$



Direct sums

1.40 Definition direct sum

Let $U_1, \dots, U_m$ be subspaces of V. Then

- $U_1 + \dots + U_m$ is a direct sum if each element of $U_1 + \dots + U_m$ is written in ONLY ONE WAY as the sum

$$u_1 + \dots + u_m$$

where $u_1 \in U_1, \dots u_m \in U_m$

- $U_1 \oplus \dots \oplus U_m$ is the NOTATION to denote the direct sum

1.44 Condition for a direct sum

Let $U_1, \dots U_m$ be subspaces of V. Then $U_1 + \dots + U_m$ is a direct sum if and only if the only way to write 0 as a sum $u_1 + \dots + u_m$, where $u_1 \in U_1, \dots, u_m \in U_m$ is by taking $u_1 = 0, \dots, u_m = 0$

Forward direction: If $U_1 + \cdots + U_m$ is a direct sum, then the only way to write 0 is taking $u_1 = 0, \ldots, u_m = 0$
[as a sum $u_1 + \cdots + u_m$]

Suppose $U_1 + \cdots + U_m$ is a direct sum. Then, by definition of the ~~a~~ direct sum, we can only write 0 as a sum $u_1 + \cdots + u_m$ in one way: by taking $u_1 = 0, \ldots, u_m = 0$

$$(0 = u_1 + \cdots + u_m)$$

Backward direction: If the only way to write 0 as a sum $u_1 + \cdots + u_m$ is by taking $u_1 = 0, \ldots, u_m = 0$, then $U_1 + \cdots + U_m$ is a direct sum

Suppose the only way to write $0 = u_1 + \cdots + u_m$ is to state $u_1 = 0, \ldots, u_m = 0$. We claim that $U_1 + \cdots + U_m$ is a direct sum. Let $v \in U_1 + \cdots + U_m$. The by 1.40, we can write

$*$ $v = u_1 + \cdots + u_m$ where $u_1 \in U_1, \ldots, u_m \in U_m$

we need to show that this representation is unique. To show this, consider ANOTHER representation

$*$ $v = v_1 + \cdots + v_m$ where $v_1 \in U_1, \ldots, v_m \in U_m$

subtract $*$ and $*$, we get

$$0 = (u_1 - v_1) + \cdots + (u_m - v_m).$$

Since we assumed in the backward direction that the only way to write 0 as a sum $u_1 + \cdots + u_m$ is to take $u_1 = 0, \ldots, u_m = 0$, we have from $* - *$ that we need to take

$$u_1 - v_1 = 0, \ldots, u_m - v_m = 0$$

so

$$u_1 = v_1, \ldots, u_m = v_m$$

So $v = v_1 + \cdots + v_m$
$= u_1 + \cdots + u_m$

Therefore, our representation of v is unique, that is, written in ONLY ONE WAY

So $U_1 + \cdots + U_m$ is a direct sum

1.45 Direct sum of two subspaces

Suppose U and W are subspaces of V. Then $U+W$ is a direct sum if and only if $U \cap W = \{0\}$

Forward direction: If $U+W$ is a direct sum, the $U \cap W = \{0\}$

Suppose $U+W$ is a direct sum. Suppose $v \in U \cap W$. Then

$$0 = v + (-v)$$

where $v \in U$ and $-v \in W$. Since $0 \in U+W$, by the definition of direct sum, we can write 0 in ONLY ONE WAY. Namely, we conclude $v=0$. Therefore $U \cap W \subset \{0\}$. At the same time, we know that $U \cap W$ is a subspace of V, which means in particular $0 \in U \cap W$, or $\{0\} \subset U \cap W$

Since $U \cap W \subset \{0\}$ and $\{0\} \subset U \cap W$, we conclude $U \cap W = \{0\}$

* Backward direction: If $U \cap W = \{0\}$, then $U+W$ is a direct sum

Suppose $U \cap W = \{0\}$ Let $u \in U$, $w \in W$ satisfy

$$0 = u + w$$

But $0 = u + w$ implies

$$u = w \in W$$

so $u \in U$ and $u \in W$; namely, $u \in U \cap W$ so $u \in \{0\}$ and so $u = 0$. And $0 = u + w$ with $u = 0$ implies $w = 0$ so the only way to write 0 as a sum $u + w$ is to take $u = 0$, $w = 0$.

* By 1.44, $U+W$ is a direct sum.