1.C Subspaces

1.32 **Definition:** ~~Let U be~~

Let V be a vector space (with addition & scalar multiplication). Let U be a subset of V. Then U is a $\boxed{\text{subspace}}$ of V if U is also a vector space with the same addition and scalar multiplication.

1.34 **Conditions of subspace**

U is a subspace of V if and only if it satisfies:

ADDITIVE Identity: $0 \in U$

Closed Under Addition: $u, w \in U$ implies $u + w \in U$

Closed Under Scalar Multiplication: $a \in \mathbb{F}$ and $u \in U$ implies $au \in U$.

$(\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C})$

1.35 Examples of Subspaces:

(a) If $b \in \mathbb{F}$, then $U_1 = \{(x_1, x_2, x_3, x_4) \in \mathbb{F}^4 \overset{\text{such that}}{:} x_3 = 5x_4 + b\}$ is a subspace of $\mathbb{F}^4$ if and only if $b = 0$.

Proof: Additive Identity: $\longrightarrow 0 \in U \longrightarrow$ is $\overset{(x_1, x_2, x_3, x_4)}{(0,0,0,0)} \in U_1$?

$x_3 = 5x_4 + b$

$0 = 5(0) + b$

$0 = b$

So additive identity is satisfies if and only if $b = 0$. (Otherwise, if $b \neq 0$, then the statement $0 = b$ will be false.

$0 = b \neq 0 \longrightarrow$ contradiction)

Closed under addition:

Let $(x_1, x_2, x_3, x_4), (y_1, y_2, y_3, y_4) \in U_1$

Then $x_3 = 5x_4 + b$ and $y_3 = 5y_4 + b$

So we have $x_3 + y_3 = (5x_4 + b) + (5y_4 + b)$

$= 5(x_4 + y_4) + 2b$

If $b = 0$, we have $x_3 = 5x_4$ and $y_3 = 5y_4$, and $x_3 + y_3 = 5(x_4 + y_4)$

So if $b = 0$, then $(x_1, x_2, x_3, x_4) + (y_1, y_2, y_3, y_4) = (x_1+y_1, x_2+y_2, x_3+y_3, x_4+y_4) \in U_1$

So if $b = 0$, then U_1 is closed under addition.

Closed under scalar multiplication:

Let $a \in \mathbb{F}$ and $(x_1, x_2, x_3, x_4) \in U_1$.

Then $a(x_3 = 5x_4 + b)a$

So we have $ax_3 = 5ax_4 + ab$

If $b = 0$, then $x = 5x_4$ and $ax_3 = 5(ax_4)$

So if $b = 0$, then $a(x_1, x_2, x_3, x_4) = (ax_1, ax_2, ax_3, ax_4) \in U_1$

So U_1 is closed under scalar multiplication.

$\therefore$ Therefore, U_1 is a subspace of V.

(b) Let U_2 be the set of continuous real-valued functions on the interval $[0,1]$. Then U_2 is a subspace of $\mathbb{R}^{[0,1]} = \{f : [0,1] \longrightarrow \mathbb{R}\}$

Additive Identity:

The zero function is continuous. So $0 \in U_2$.

Closed under Addition:

Let $f, g \in U_2$ (Let $f, g : [0,1] \rightarrow \mathbb{R}$ be continuous)

Sum of two continuous functions is continuous.

So $f + g \in U_2$

Closed Under scalar multiplication

Let $a \in \mathbb{F}$, $f \in U_2$. (Let $f : [0,1] \longrightarrow \mathbb{R}$ be continuous.)

Scalar ~~multiplication~~ multiple of continuous functions is continuous.

So $af \in U_2$.

So U_2 is a subspace of $\mathbb{R}^{[0,1]}$.

Sums of Subspace

1.36 Definition: Sum of subsets

Let $U_1, \dots, U_m$ be subsets of V. The sum of $U_1, \dots, U_m$ is denoted $U_1 + \dots + U_m$ and is the set of all possible sums of elements of $U_1, \dots, U_m$.

In other words, $U_1 + \dots + U_m = \{u_1 + \dots + u_m : u_1 \in U_1, \dots, u_m \in U_m\}$

1.37 Example. Let $\mathbb{F}^3$ be a vector space.

Let $U = \{(x,0,0) \in \mathbb{F}^3 : x \in \mathbb{F}\}$

and $W = \{(0,y,0) \in \mathbb{F}^3 : y \in \mathbb{F}\}$

Then $U+W = \{(x,y,0) : x, y \in \mathbb{F}\}$.

$(x,0,0) \in U$

$(0,y,0) \in W$

$$\underbrace{(x,0,0)}_{\in U} + \underbrace{(0,y,0)}_{\in W} \in \underbrace{U+W}_{V}$$

$= (x+0, 0+y, 0+0) \in U+W$

$= (x, y, 0) \in U+W$

1.39 Sum of Subspaces is the smallest containing subspace. ※

Suppose $U_1, \dots, U_m$ are subspaces of V. Prove that $U_1 + \cdots + U_m$ is the SMALLEST subspace of V containing $U_1, \dots, U_m$.

Proof: First, we ~~contain~~ claim that $U_1 + \cdots + U_m$ is a subspace of V.

Since $U_1, \dots, U_m$ are subspaces of V, we have:

Additive Identity: $0 \in U_1, \dots, 0 \in U_m$

Closed Under Addition: $u_1, w_1 \in U_1, \dots, u_m, w_m \in U_m$ imply $u_1 + w_1 \in U_1, \dots, u_m + w_m \in U_m$.

Closed Under Scalar Multiplication: $a \in \mathbb{F}$, $u_1 \in U_1, \dots, u_m \in U_m$ imply $au_1 \in U_1, \dots, au_m \in U_m$.

~~So~~ we have $0 = \underbrace{0 + \cdots + 0}_{m} \in U_1 + \cdots + U_m$

So $U_1 + \cdots + U_m$ contains the additive identity.

We also have: If $u, w \in U_1 + \cdots + U_m$ then we can write

$u = u_1 + \cdots + u_m$ and $w = w_1 + \cdots + w_m$.

$$\begin{aligned} \text{So } u + w &= (u_1 + \cdots + u_m) + (w_1 + \cdots + w_m) \\ &= (u_1 + w_1) + \cdots + (u_m + w_m) \in U_1 + \cdots + U_m. \end{aligned}$$

Therefore, $U_1 + \cdots + U_m$ is closed under addition.

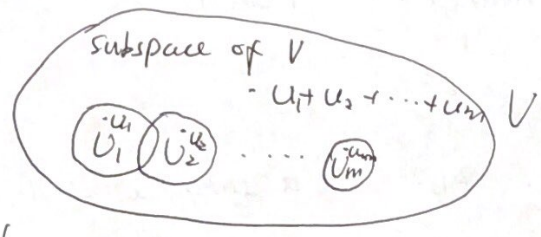
Let $a \in \mathbb{F}$. We also have $au = a(u_1 + \cdots + u_m)$
$= au_1 + \cdots + au_m \in U_1 + \cdots + U_m$

Therefore, $U_1 + \cdots + U_m$ is closed under scalar multiplication.

So $U_1 + \cdots + U_m$ a subspace of V.

Next, we claim that $U_1 + \cdots + U_m$ is the SMALLEST subspace of V containing $U_1, \ldots, U_m$.

Let $i = 1, \ldots, m$. Let $x_i \in U_i$. Then $x_i = \underbrace{(0 + \cdots + 0)}_{i-1 \text{ terms}} + \underbrace{x_i}_{i\text{th term}} + \underbrace{(0 + \cdots + 0)}_{m-i \text{ terms}}$



$\in U_1 + \cdots + U_{i-1} + U_i + U_{i+1} + \cdots + U_m$
$= U_1 + \cdots + U_m$

Observe that every subspace of V that contains $U_1, \ldots, U_m$ must also contain finite sums of elements of $U_1, \ldots, U_m$
(This is due to the property of closed under addition for subspaces.)
This means in particular that every subspace that contains $U_1, \ldots, U_m$ must contain the subspace $U_1 + \cdots + U_m$.
Since $U_1 + \cdots + U_m$ is contained in <u>every</u> subspace that contains $U_1, \ldots, U_m$.
So $U_1 + \cdots + U_m$ is the SMALLEST subspace that contains $U_1, \ldots, U_m$.

Direct Sums

1.40 Definition: direct sum

Let $U_1, \ldots, U_m$ be subspaces of V. Then $U_1 + \cdots + U_m$ is a direct sum if each element of $U_1 + \cdots + U_m$ is written in only one way as the sum $u_1 + \cdots + u_m$, where $u_1 \in U_1, \ldots, u_m \in U_m$.

- $U_1 \oplus \cdots \oplus U_m$ is the NOTATION to denote the direct sum.

1.44 Condition of a direct sum

Let $U_1, \ldots, U_m$ be subspaces of V. Then $U_1 + \cdots + U_m$ is a direct sum $\boxed{\text{if and only if}}$ the only way to write 0 as a sum $u_1 + \cdots + u_m$, where $u_1 \in U_1, \ldots, u_m \in U_m$, is by taking $u_1 = 0, \ldots, u_m = 0$.

Proof:

Forward direction: If $U_1 + \cdots + U_m$ is a direct sum, then the only way to write 0 as a sum $u_1 + \cdots + u_m$ is taking $u_1 = 0, \ldots, u_m = 0$.

Suppose $U_1 \oplus \cdots \oplus U_m$ is a direct sum.

Then, by definition of the direct sum, we can only write 0 as a sum $u_1 + \cdots + u_m$ ($0 = u_1 + \cdots + u_m$) in one way: by taking $u_1 = 0, \ldots, u_m = 0$

Backward direction: If the only way to write 0 as a sum $u_1 + \cdots + u_m$ is by taking $u_1 = 0, \ldots, u_m = 0$, then $U_1 + \cdots + U_m$ is a direct sum.

Suppose the only way to write $0 = u_1 + \cdots + u_m$ is to take $u_1 = 0, \ldots, u_m = 0$

We claim that $U_1 + \cdots + U_m$ is a direct sum. Let $v \in U_1 + \cdots + U_m$.

Then by 1.40, we can write ☆ $v = u_1 + \cdots + u_m$ where $u_1 \in U_1, \ldots, u_m \in U_m$

We need to show that this representation is unique. To show this, consider ANOTHER representation ☆ $v = v_1 + \cdots + v_m$ where $v_1 \in U_1, \ldots, v_m \in U_m$

subtract ☆ and ☆, we get $0 = (v_1 - u_1) + \cdots + (v_m - u_m)$.

Since we assumed in the backward direction that the only ~~to~~ way to write 0 as a sum $u_1 + \dots + u_m$ is to take $u_1 = 0, \dots, u_m = 0$, we have from ☆ − ☆ that we need to take

$$u_1 - v_1 = 0, \quad \dots\dots, \quad u_m - v_m = 0$$

So $\quad u_1 = v_1, \dots, u_m = v_m$

So $\quad v = v_1 + \dots + v_m$
$\qquad = u_1 + \dots + u_m$

Therefore, our representation of v is unique, that is, written in only ONE way.

So $U_1 + \dots + U_m$ is a direct sum.

1.45 Direct sum of two spaces.

Suppose U and W are subspaces of V. Then $U+W$ is a direct sum if and only if $U \cap W = \{0\}$.

Forward direction: If $U+W$ is a direct sum, then $U \cap W = \{0\}$.

Suppose $U+W$ is a direct sum. Suppose $v \in U \cap W$

Then $\quad 0 = v + (-v)$,

where $v \in U$ and $-v \in W$. Since $0 \in U+W$, by the definition of direct sum, we can write 0 in ONLY ONE way.

Namely, we conclude $v = 0$. Therefore, $U \cap W \subset \{0\}$.

At the same time, we know that $U \cap W$ is a subspace of V, which means in particular $0 \in U \cap W$, or $\{0\} \subset U \cap W$.

Since $U \cap W \subset \{0\}$ and $\{0\} \subset U \cap W$, we conclude $U \cap W = \{0\}$.

Backward direction: If $U \cap W = \{0\}$, then $U+W$ is a direct sum.

Suppose $U \cap W = \{0\}$. Let $u \in U$, $w \in W$ satisfy $0 = u+w$.

But $0 = u+w$ implies $u = -w \in W$.

So $u \in U$ and $u \in W$; namely, $u \in U \cap W$.

So $u \in \{0\}$, and so $u = 0$.

And $0 = u+w$ with $u=0$ implies $w=0$.

So the only way to write 0 as a sum $u+w$ is to take $u=0, w=0$

By 1.44, $U+W$ is a direct sum.