Math 131

Edward Choi

$\boxed{7/1}$ * Possible Problems on Group Exam #2:

2.A 1, 6, 7, 8, 9, 10

2.B 6, 8

2.C 16

2.A Span & Linear Independence

$\boxed{\text{Def}}$ 2.3

A <u>linear combination</u> of a list $v_1, \ldots, v_m \in V$ is a vector of the form:

$$a_1 v_1 + \ldots + a_m v_m$$

for some $a_1, \ldots, a_m \in \mathbb{F}$

Ex: Let $V = \mathbb{F}^3$. Is $(17, -4, 2)$ a linear combi? of the list $(2, 1, -3), (1, -2, 4)$

If yes then there exists $a_1, a_2 \in \mathbb{F}$ such that

$$(17, -4, 2) = a_1 (2, 1, -3) + a_2 (1, -2, 4)$$
$$= (2a_1, a_1, -3a_1) + (a_2, -2a_2, 4a_2)$$

$17 = 2a_1 + a_2$

$-4 = a_1 - 2a_2$ $\quad -8 = 2a_1 - 4a_2 \quad 25 = 5a_2 = 5 = a_2$

$2 = -3a_1 + 4a_2$ $\quad -4 = a_1 - 10 \quad a_1 = 6$

$2 \stackrel{?}{=} -18 + 20$

$2 = 2$ ✓

Therefore, $(17, -4, 2)$ is a linear combination of $(2, 1, -3), (1, -2, 4)$ but $(17, -4, 5)$ is NOT.

2.5 $\boxed{\text{Def}}$

The <u>span</u> of $v_1, \ldots v_m$ is the set of all linear combinations of $v_1, \ldots, v_m \in V$, and is denoted:

$$\text{span}(v_1, \ldots v_m) = \{a_1 v_1 + \ldots + a_m v_m : a_1, \ldots, a_m \in \mathbb{F}\}$$

$\uparrow$ linear combi. of $v_1, \ldots, v_m$

$\boxed{\text{Note}}$ (Span of the empty list () is defined to be $\{0\}$.)

2.6 Example.

$(17,-4,2) \in \text{span}[(2,1,-3),(1,-2,4)]$

$(17,-4,5) \notin \text{span}[(2,1,-3),(1,-2,4)]$

$\boxed{2.7}$ Span is the smallest containing subspace

The span of a list of vectors $v_1, \dots v_m \in V$ is the smallest subspace of V combining $v_1, \dots v_m$.

* Proof: First, we will prove that $\text{span}(v_1, \dots, v_m)$ is a subspace of V.

- Add Identi. $0 = 0v_1 + \dots + 0v_m \in \text{span}(v_1, \dots, v_m)$ ✓
- Closed under add. let $a_1v_1 + \dots + a_mv_m, c_1v_1 + \dots + c_mv_m \in \text{span}$ " "
  for some $a_1, \dots, a_m, c_1, \dots, c_m \in \mathbb{F}$, Then we have $(a_1v_1 + \dots + a_mv_m + c_1v_1 + \dots + c_mv_m)$
  $= (a_1 + c_1)v_1 + \dots + (a_m + c_m)v_m \in \text{span}(v_1, \dots v_m)$ ✓
- Closed under multi.
  let $\lambda \in \mathbb{F}$ be arbitrary, then
  $\lambda(a_1v_1 + \dots + a_mv_m) = (\lambda a_1)v_1 + \dots + (\lambda a_m)v_m \in \text{span}(v_1, \dots v_m)$

Therefore, $\text{span}(v_1, \dots, v_m)$ is a subspace of V.

Now prove it is the smallest:

First notice that each v_i $(i = 1, \dots m)$ can be written as a linear combi. of $v_1, \dots, v_m$:

$v_i = 0v_1 + \dots + 0v_{i-1} + 1v_i + 0v_{i+1} + \dots + 0v_m \in \text{span}(v_1, \dots v_m)$

In other words, $\text{span}(v_1 \dots v_m)$ contains each v_i, or equivalently $\text{span}(v_1, \dots v_m)$ contains $v_1, \dots v_m$.

Because every subspace of V is closed under scalar multi, and add, every subspace containing v_i contains a linear combi of $v_1, \dots v_m$. In other words every subspace contains $\text{span}(v_1, \dots v_m)$.

This makes $\text{span}(v_1, \dots, v_m)$ the smallest subspace of V

∎

2.8 Def:

If we have $\text{span}(v_1, \ldots, v_m) = V$,

then we say $v_1, \ldots, v_m$ spans V.

2.9 Example

$(1,0,0), (0,1,0), (0,0,1)$ spans $\mathbb{F}^3$

Proof: Let $(x_1, x_2, x_3) \in \mathbb{F}^3$ be arbitrary,

then we can write:

$$(x_1, x_2, x_3) = (x_1, 0, 0) + (0, x_2, 0) + (0, 0, x_3)$$
$$= x_1(1,0,0) + x_2(0,1,0) + x_3(0,0,1)$$

So $(x_1, x_2, x_3) \in \text{Span}((1,0,0), (0,1,0), (0,0,1))$

In other words $(1,0,0), (0,1,0), (0,0,1)$ spans $\mathbb{F}^3$

2.11 Def

- A function $p: \mathbb{F} \to \mathbb{F}$ is a polynomial if there exists $a_0, \ldots, a_m \in \mathbb{F}$ such that

$$p(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \ldots + a_m z^m \quad \forall z \in \mathbb{F}$$

- Set of all polynomials with coefficients in $\mathbb{F}$ is called $P(\mathbb{F})$.

2.12 A polymial $p \in P(\mathbb{F})$ is said to be degree m if there exists $a_0, a_1, \ldots, a_m \in \mathbb{F}$ with $a_m \neq 0$ such that

$$p(z) = a_0 + a_1 z + \ldots + a_m z^m \quad \forall z \in \mathbb{F}$$

2.10 Def: A vector space V is called finite dimensional if there exists a list $v_1, \ldots, v_m \in V$ that spans V;

$\text{spans}(v_1, \ldots v_m) = V$

2.15 Def: A vector space V is called infinite dimensional if it is Not finite dimensional.

Linear Independence

2.17 $\boxed{\text{Def}}$

A list $v_1, \ldots, v_m \in V$ is called linearly independent, if and only if the choice of $a_1, \ldots, a_m \in \mathbb{F}$ that satisfies $a_1 v_1 + \ldots + a_m v_m = 0$
is $a_1 = 0, \ldots, a_m = 0$.

2.19 $\boxed{\text{Def}}$

A list $v_1, \ldots, v_m \in V$ is linearly dependent if $a_1, \ldots a_m \in \mathbb{F}$, not all zero such that $a_1 v_1 + \ldots a_m v_m = 0$
In other words, $v_1, \ldots v_m \in V$ is NOT linearly independent.

2.18 • A list of one vector V is linearly independent IFF $v_1 \neq 0$, then if $a_1 \in \mathbb{F}$ satisfies:

$$a_1 v_1 = 0,$$
$$\text{then} \quad a_1 = 0$$

• A list $(1,0,0,0), (0,1,0,0), (0,0,1,0)$ is linearly independent in $\mathbb{F}^4$

Suppose $a_1, a_2, a_3, a_4 \in \mathbb{F}$ satisfy

$$a_1(1,0,0,0) + a_2(0,1,0,0) + a_3(0,0,1,0) = (0,0,0,0)$$
$$(a_1,0,0,0) + (0,a_2,0,0) + (0,0,a_3,0) = (0,0,0,0)$$
$$(a_1, a_2, a_3, 0) = (0,0,0,0)$$
$$a_1 = 0 \ \checkmark$$
$$a_2 = 0 \ \checkmark$$
$$a_3 = 0 \ \checkmark$$

2.20 Example

$(2,3,1), (1,-1,2), (7,3,8)$ is linearly dependent

Suppose $a_1, a_2, a_3 \in \mathbb{F}$ satisfy

$$a_1(2,3,1) + a_2(1,-1,2) + a_3(7,3,8) = (0,0,0)$$
$$(2a_1, 3a_1, a_1) + (a_2, -a_2, 2a_2) + (7a_3, 3a_3, 8a_3) = (0,0,0)$$

$2a_1 + a_2 + 7a_3 = 0 \quad 3a_1 - a_2 + 3a_3 = 0 \quad a_1 + 2a_2 + 8a_3 = 0$

$a_1 = 2 \quad a_2 = 3 \quad a_3 = 1$

So at least one scalar is non-zero.

So $(2,3,1),(1,-1,2),(7,3,8)$ is linearly dependent in $\mathbb{F}^3$

- Every list of vectors in V containing the zero vector such as $v_1, v_2, v_3, v_4, v_5, v_6$, is linearly dependent.

Suppose $a_1, a_2, \ldots, a_6 \in \mathbb{F}$ satisfy

$$a_1 v_1 + a_2 v_2 + a_3 (0) + a_4 v_4 + a_5 v_5 + a_6 v_6 = 0$$

then set

$$a_1 = 0, a_2 = 0, a_3 = 1, a_4 = 0, a_5 = 0, a_6 = 0$$

So the scalar are NOT ALL zero.