Possible problems on Group exam 2:

2A: 1, 6 7 8 9 10

2B: 6, 8

2C: 16.

2A. Span and Linear Independence.

2.3. Definition:

A linear combination of a list $v_1, \dots, v_m \in V$ is a vector of the form:

$$a_1 v_1 + \dots + a_m v_m \quad \text{for some } a_1, \dots, a_m \in \mathbb{F}.$$

Example:

Let $V = \mathbb{F}^3$. Is $(17, -4, \overset{5}{2})$ a linear combination of the list $(2, 1, 3)$, $(1, -2, 4)$?

If yes, then there exist $a_1, a_2 \in \mathbb{F}$ such that,

$$(17, -4, \overset{5}{2}) = a_1(2, 1, -3) + a_2(1, -2, 4)$$

That is, we have

$$(17, -4, \overset{5}{2}) = (2a_1, a_1, -3a_1) + (a_2, -2a_2, 4a_2)$$
$$= (2a_1 + a_2,\ a_1 - 2a_2,\ -3a_1 + 4a_2)$$

$$\Rightarrow \begin{cases} 17 = 2a_1 + a_2 & \text{①} \\ -4 = a_1 - 2a_2 & \text{②} \\ \underset{5}{2} = -3a_1 + 4a_2 & \text{③} \end{cases} \quad \text{①②} \Rightarrow a_1 = 6 \quad a_2 = 5$$

③ $\Rightarrow$ $2 \overset{?}{=} -3\times 6 + 4\times 5$ ✓

$5 \neq -3\times 6 + 4\times 5$ ✗

Therefore, $(17, -4, 2)$ is a linear combination of $(2, 1, -3)$, $(1, -2, 4)$, but $(17, -4, 5)$ is not.

2.5 Definition

The span of $v_1, \dots, v_m$ is the set of all linear combina-tions of $v_1, \dots, v_m \in V$, and is denoted:

$$\operatorname{span}(v_1, \dots, v_m) = \{\underbrace{a_1v_1 + \cdots + a_mv_m}_{\text{linear combination of } v_1, \dots, v_m} : a_1, \dots, a_m \in \mathbb{F}\}$$

(The span of the empty list ( ) is defined to be $\{0\}$)

2.6 Example:

In $V = \mathbb{F}^3$,

- $(17, -4, 2)$ is a linear combination of $(2, 1, -3)$, $(1, -2, 4)$

  Therefore, $(17, -4, 2) \in \operatorname{span}((2, 1, -3), (1, -2, 4))$

- $(17, -4, 5)$ is not a linear combination of $(2, 1, -3)$ $(1, -2, 4)$.

  Therefore, $(17, -4, 5) \notin \operatorname{span}((2, 1, -3), (1, -2, 4))$.

2.7

Span is the smallest containing subspace

The span of a list of vectors $v_1, \dots, v_m \in V$ is the smallest subspace of V containing $v_1, \dots, v_m$.

Proof: First, we will prove that $\operatorname{span}(v_1, \dots, v_m)$ is a subspace of V.

- Additive identity: $0 = 0v_1 + \cdots + 0v_m \in \operatorname{Span}(v_1, \dots, v_m)$
- Closed under addition: Let $a_1v_1 + \cdots + a_mv_m, c_1v_1, \dots, c_mv_m \in \operatorname{Span}(v_1, \dots, v_m)$ for some $a_1, \dots, a_m, c_1, \dots, c_m \in \mathbb{F}$.

  Then we have $(a_1v_1 + \cdots + a_mv_m) + (c_1v_1 + \cdots + c_mv_m)$

$= ((a_1+c_1)v_1 + \cdots + (a_m+c_m)v_m) \in \text{Span}(v_1, \ldots, v_m)$

- Closed under scalar multiplication:

  Let $\lambda \in \mathbb{F}$ be arbitrary. Then

  $\lambda(a_1v_1 + \cdots + a_mv_m) = (\lambda a_1)v_1 + \ldots + (\lambda a_m)v_m \in \text{Span}(v_1 \ldots v_m)$

Therefore, $\text{span}(v_1, \cdots, v_m)$ is a subspace of V.

Now, we will prove that $\text{span}(v_1, \ldots, v_m)$ is the smallest subspace of V.

First, note that each v_j $(j = 1, \cdots, m)$ can be written as a linear combination of $v_1, \ldots, v_m$

$$v_j = 0v_1 + \cdots + 0v_{j-1} + 1v_j + 0v_{j+1} + \cdots + 0v_m \in \text{span}(v_1, \cdots, v_m)$$

In other words, $\text{span}(v_1, \ldots, v_m)$ contains each v_j, or equally $\text{span}(v_1, \ldots, v_m)$ contains $v_1, \ldots, v_m$

Also, because every subspace of V is closed under scalar and addition, every subspace containing v_j contains all linear combination of $v_1, \ldots, v_m$. In other words, every subspace contains $\text{span}(v_1, \ldots, v_m)$

This makes $\text{span}(v_1, \ldots, v_m)$ the smallest subspace of V.

2.8 Definition

If we have $\text{span}(v_1, \ldots, v_m) = V$, the we say $v_1, \ldots, v_m$ **spans** V.

2.9 Example

The list $(1,0,0), (0,1,0), (0,0,1)$ spans $\mathbb{F}^3$.

Proof: Let $(x_1, x_2, x_3) \in \mathbb{F}^3$ be arbitrary.

Then we can write:

$$\begin{aligned}(x_1, x_2, x_3) &= (x_1, 0, 0) + (0, x_2, 0) + (0, 0, x_3) \\ &= x_1(1,0,0) + x_2(0,1,0) + x_3(0,0,1)\end{aligned}$$

So, $(x_1, x_2, x_3) \in \operatorname{Span}\{(1,0,0), (0,1,0), (0,0,1)\}$

In other words, $(1,0,0), (0,1,0), (0,0,1)$ spans $\mathbb{F}^3$.

2.10 Definition

- A function $p: \mathbb{F} \to \mathbb{F}$ is a polynomial if there exist $a_0, \dots, a_m \in \mathbb{F}$ such that

$$p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_m z^m$$

for all $z \in \mathbb{F}$.

- The set of all polynomials with coefficients in $\mathbb{F}$ is called $P(\mathbb{F})$.

2.12 A polynomial $p \in P(\mathbb{F})$ is said to have degree m if there exist $a_0, a_1, \cdots, a_m \in \mathbb{F}$ with $a_m \neq 0$ such that

$p(z) = a_0 + a_1 z + \cdots + a_m z^m$ for all $z \in \mathbb{F}$.

Definition

A vector space V is called finite-dimensional if there exists a list $v_1, \dots, v_m \in V$ that spans V; that is.

$$\operatorname{span}(v_1, \dots, v_m) = V.$$

2.15 Definition

A vector space V is called infinite-dimensional if it is not finite dimensional.

Linear Independence.

2.17 Definition

- A list $v_1, \dots, v_m \in V$ is called linearly independent if the only choice of $a_1, \dots, a_m \in \mathbb{F}$ that satisfies

$$a_1 v_1 + \dots + a_m v_m = 0$$

is $a_1 = 0, \dots, a_m = 0$.

2.19 Definition

A list $v_1, \dots, v_m \in V$ is linearly dependent if there exist $a_1, \dots, a_m \in \mathbb{F}$, not all zero such that

$$a_1 v_1 + \dots + a_m v_m = 0$$

In other words, $v_1 \dots v_m \in V$ is not linearly independent.

2.18 Example Linearly independent lists

- A list v_1 of one vector $v_1 \in V$ is linearly independent iff $v_1 \neq 0$. Then if $a_1 \in \mathbb{F}$ satisfies.

$$a_1 v_1 = 0,$$

then $a_1 = 0$.

- A list $(1,0,0,0)$, $(0,1,0,0)$, $(0,0,1,0)$ is linearly independent in $\mathbb{F}^4$

Suppose $a_1, a_2, a_3, a_4 \in \mathbb{F}$ satisfy

$a_1(1,0,0,0) + a_2(0,1,0,0) + a_3(0,0,1,0) = (0,0,0,0)$

Then we have

$$(a_1, 0, 0, 0) + (0, a_2, 0, 0) + (0, 0, a_3, 0) = (0,0,0,0)$$

$$(a_1, a_2, a_3, 0) = (0,0,0,0)$$

Equate the coordinates:

$$a_1 = 0, \quad a_2 = 0, \quad a_3 = 0 \ ,$$

2.20 Example. Linearly dependent lists.

- $(2,3,1), (1,-1,2), (7,3,8)$ is linearly dependent in $\mathbb{F}^3$.

suppose $a_1, a_2, a_3 \in \mathbb{F}$ satisfy

$$a_1(2,3,1) + a_2(1,-1,2) + a_3(7,3,8) = (0,0,0)$$

$$(2a_1, 3a_1, a_1) + (a_2, -a_2, 2a_2) + (7a_3, 3a_3, 8a_3) = (0,0,0)$$

$$\Rightarrow (2a_1 + a_2 + 7a_3, 3a_1 - a_2 + 3a_3, a_1 + 2a_2 + 8a_3) = (0,0,0)$$

$$\begin{cases} 2a_1 + a_2 + 7a_3 = 0 \\ 3a_1 - a_2 + 3a_3 = 0 \\ a_1 + 2a_2 + 8a_3 = 0 \end{cases} \Rightarrow \begin{cases} a_1 = 2 \\ a_2 = 3 \\ a_3 = -1 \end{cases}$$

So at least one of scalars is non-Zero

So $(2,3,1), (1,-1,2), (7,3,8)$ is linearly dependent in $\mathbb{F}^3$.

- Every list of vectors in V containing the zero vector, such as $v_1, v_2, 0, v_4, v_5, v_6$ is linearly dependent.

Suppose $a_1, a_2, a_3, a_4, a_5, a_6 \in \mathbb{F}$, satisfy.

$$a_1 v_1 + a_2 v_2 + a_3 \cdot 0 + a_4 v_4 + a_5 v_5 + a_6 v_6 = 0$$

Then set

$$a_1 = 0, a_2 = 0, a_3 = 1, a_4 = 0, a_5 = 0, a_6 = 0$$

So the scalars are not all zero.

$\therefore$ linearly dependent.