7/1 Office Hours
MTWR 11:10 - 12pm
Skye 277

*Possible problems from HW 2 on orap Exam 2
2.A. 1, 6, 7, 8, 9, 10
2.B. 6, 8
2.C. 16

Section 2.A. Span and Linear Independence

2.3 Definition:
A <u>linear combination</u> of a list $v_1, \ldots, v_m \in V$ is a vector of the form
$$a_1 v_1 + \ldots + a_m v_m$$
for some $a_1, \ldots, a_m \in \mathbb{F}$

Example: Let $V = \mathbb{F}^3$. Is $(17, -4, 2)$ a linear combination of the list $(2, 1, -3), (1, -2, 4)$?

If yes, then there exist $a_1, a_2 \in \mathbb{F}$ such that
$$(17, -4, 2) = a_1 (2, 1, -3) + a_2 (1, -2, 4)$$
That is, we have
$$(17, -4, 2) = (2a_1, a_1, -3a_1) + (a_2, -2a_2, 4a_2)$$
$$= (2a_1 + a_2, a_1 - 2a_2, -3a_1 + 4a_2)$$

$17 = 2a_1 + a_2$ ①
$-4 = a_1 - 2a_2$ ②
$2 = -3a_1 + 4a_2$

① - 2 ②

$17 = 2a_1 + a_2$
$-2(-4 = a_1 - 2a_2)$ $\Rightarrow$ $+\begin{array}{l} 17 = 2a_1 + a_2 \\ 8 = -2a_1 + 4a_2 \end{array}$

$25 = 5a_2$

$a_2 = 5$

plug a_2 into ①

$17 = 2a_1 + 5$

$12 = 2a_1$

$a_1 = 6$

In other words, every subspace of V that contains $v_1, \dots, v_m \in V$ must also contain span$(v_1, \dots, v_m)$. This makes span$(v_1, \dots, v_m)$ the SMALLEST subspace of V that contains $v_1, \dots, v_m$.

plug a_1 & a_2 into ③

$2 = -3a_1 + 4a_2$

$2 = -3(6) + 4(5)$

$2 = -18 + 20$

$2 = 2$ ✓

If $(17, -4, 5)$ then $5 \neq -3(6) + 4(5)$.
Therefore $(17, -4, 2)$ is a linear combination of $(2, 1, -3), (1, -2, 4)$, BUT $(17, -4, 5)$ is NOT a linear combination of $(2, 1, -3), (1, -2, 4)$.

Proof of 2.7 statement: Let U be a subspace of V that contains $v_1, \dots, v_m$. Since subspaces are closed under addition and scalar multiplication, $v_1, \dots, v_m \in U$ implies $a_1v_1 + \dots + a_mv_m \in U$ for all $a_1, \dots, a_m \in \mathbb{F}$. By def, span$(v_1, \dots, v_m) = \{a_1v_1 + \dots + a_mv_m : a_1, \dots, a_m \in \mathbb{F}\}$.
Therefore, span$(v_1, \dots, v_m) \subset U$.

2.5 Definition:
The span of $v_1, \ldots, v_m$ is the set of all linear combinations of $v_1, \ldots, v_m \in V$, and is denoted

$$\text{span}(v_1, \ldots, v_m) = \{\underbrace{a_1v_1 + \ldots + a_mv_m}_{\text{linear combination of } v_1, \ldots, v_m} \in V : a_1, \ldots, a_m \in \mathbb{F}\}$$

(The span of the empty list ( ) is defined to be $\{0\}$.)

2.6 Example

- In $V = \mathbb{F}^3$, $(17, -4, 2)$ is a linear combination of $(2,1,-3), (1,-2,4)$
  Therefore, $(17, -4, 2) \in \text{span}((2,1,-3), (1,-2,4))$
- $(17, -4, 5)$ is NOT a linear combination of $(2,1,-3), (1,-2,4)$.
  Therefore, $(17, -4, 5) \notin \text{span}((2,1,-3), (1,-2,4))$

2.7 Span is the smallest containing subspace

The span of a list of vectors $v_1, \ldots, v_m \in V$ is the smallest subspace of V containing $v_1, \ldots, v_m$.

Proof: First, we will prove that $\text{span}(v_1, \ldots, v_m)$ is a subspace of V. Every subspace of V containing each v_j; $j = 1, \ldots, m$ ~~$v_1, \ldots, v_m$~~ contains $\text{span}(v_1, \ldots, v_m)$.

- Additive Identity: $0 = 0v_1 + \ldots + 0v_m \in \text{span}(v_1, \ldots, v_m)$
- Closed under Addition: → In other words, if a subspace contains $v_1, \ldots, v_m$, then it contains $\text{span}(v_1, \ldots, v_m)$

Let $a_1v_1 + \ldots + a_mv_m$, $c_1v_1 + \ldots + c_mv_m \in \text{span}(v_1, \ldots, v_m)$ for some $a_1, \ldots, a_m, c_1, \ldots, c_m \in \mathbb{F}$. Then we have

$$(a_1v_1 + \ldots + a_mv_m) + (c_1v_1 + \ldots + c_mv_m)$$
$$= (a_1 + c_1)v_1 + \ldots + (a_m + c_m)v_m \in \text{span}(v_1, \ldots, v_m)$$

• Closed under scalar multiplication:
Let $\lambda \in \mathbb{F}$ be arbitrary. Then
$$\lambda(a_1 v_1 + \dots + a_m v_m) = (\lambda a_1)v_1 + \dots + (\lambda a_m)v_m$$
$$\in \text{span}(v_1, \dots, v_m)$$

Therefore, span$(v_1, \dots, v_m)$ is a subspace of V.

Now, we will prove that span$(v_1, \dots, v_m)$ is the <u>SMALLEST</u> subspace of V.

First, notice that each v_j (for all $j = 1, \dots, m$) can be written as a linear combination of $v_1, \dots, v_m$:
$$v_j = 0v_1 + \dots + 0v_{j-1} + 1v_j + 0v_{j+1} + \dots + 0v_m$$
$$\in \text{span}(v_1, \dots, v_m)$$

In other words, span$(v_1, \dots, v_m)$ contains each v_j, or equivalently span$(v_1, \dots, v_m)$ contains $v_1, \dots, v_m$.

Because every subspace of V is closed under scalar multiplication and addition, every subspace containing v_j contains all linear combination of $v_1, \dots, v_m$.

In other words, every subspace contains span$(v_1, \dots, v_m)$.

This makes span$(v_1, \dots, v_m)$ the SMALLEST subspace of V. ∎

2.8 Definition:

If we have $\text{span}(v_1, \ldots, v_m) = V$, then we say $v_1, \ldots, v_m$ spans V.

2.9 Example

The list
$(1,0,0), (0,1,0), (0,0,1)$ spans $\mathbb{F}^3$.

Proof: Let $(x_1, x_2, x_3) \in \mathbb{F}^3$ be arbitrary.
Then we can write

$$(x_1, x_2, x_3) = (x_1, 0, 0) + (0, x_2, 0) + (0, 0, x_3)$$
$$= x_1(1,0,0) + x_2(0,1,0) + x_3(0,0,1)$$

So, ~~[illegible]~~ $(x_1, x_2, x_3) \in \text{span}\big((1,0,0), (0,1,0), (0,0,1)\big)$.

In other words, $(1,0,0), (0,1,0), (0,0,1)$ spans $\mathbb{F}^3$.

2.10 Definition

- A function $p: \mathbb{F} \to \mathbb{F}$ is a <u>polynomial</u> if there exist $a_0, \ldots, a_m \in \mathbb{F}$ such that
$$p(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \ldots + a_m z^m$$
for all $z \in \mathbb{F}$.
- The set of all polynomials with coefficients in $\mathbb{F}$ is called $\mathcal{P}(\mathbb{F})$.

2.12 A polynomial $p \in \mathcal{P}(\mathbb{F})$ is said to ~~b~~ have degree m if there exist $a_0, a_1, \ldots, a_m \in \mathbb{F}$ with $a_m \neq 0$ such that

$$p(z) = a_0 + a_1 z + \ldots + a_m z^m$$

for all $z \in \mathbb{F}$.

2.10 Definition:

A vector space V is called finite-dimensional if there exists a list $v_1, \ldots, v_m \in V$ that spans V, that is

$$\text{Span}(v_1, \ldots, v_m) = V.$$

2.15 Definition:

A vector space V is called infinite-dimensional if it is NOT finite dimensional.

Linear Independence

1

2.17 Definition:

- A list $v_1, \ldots, v_m \in V$ is called linearly independent if the only choice of $a_1, \ldots, a_m \in \mathbb{F}$ that satisfies

$$a_1 v_1 + \ldots + a_m v_m = 0$$

is $a_1 = 0, \ldots, a_m = 0$.

2.19 Definition:

- A list $v_1, \ldots, v_m \in V$ is linearly dependent if there exist $a_1, \ldots, a_m \in \mathbb{F}$, not all zero (in other words, some nonzero) such that $a_1 v_1 + \ldots + a_m v_m = 0$.

In other words, $v_1, \ldots, v_m \in V$ is NOT linearly independent.

2.18 Example | Linearly Independent Lists

- A list v_1 of one vector $v_1 \in V$ is linearly independent if and only if $v_1 \neq 0$. Then if $a_1 \in \mathbb{F}$ satisfies
$$a_1 v_1 = 0,$$
then
$$a_1 = 0.$$

- A list $(1,0,0,0), (0,1,0,0), (0,0,1,0)$ is linearly independent in $\mathbb{F}^4$.

Suppose $a_1, a_2, a_3, a_4 \in \mathbb{F}$ satisfies
$$a_1(1,0,0,0) + a_2(0,1,0,0) + a_3(0,0,1,0) = (0,0,0,0)$$
Then we have $(a_1,0,0,0) + (0,a_2,0,0) + (0,0,a_3,0) = (0,0,0,0)$
$$(a_1,a_2,a_3,0) = (0,0,0,0)$$
Equate the coordinates.
$$a_1 = 0,\ a_2 = 0,\ a_3 = 0 \quad ✓$$

2.20 Example | Linearly Dependent Lists

- $(2,3,1), (1,-1,2), (7,3,8)$ is linearly dependent in $\mathbb{F}^3$

Suppose $a_1, a_2, a_3 \in \mathbb{F}$ satisfy
$$a_1(2,3,1) + a_2(1,-1,2) + a_3(7,3,8) = (0,0,0)$$
$$(2a_1, 3a_1, a_1) + (a_2, -a_2, 2a_2) + (7a_3, 3a_3, 8a_3) = (0,0,0)$$
$$(2a_1 + a_2 + 7a_3,\ 3a_1 - a_2 + 3a_3,\ a_1 + 2a_2 + 8a_3) = (0,0,0)$$

$2a_1 + a_2 + 7a_3 = 0$ system solve $\Rightarrow$ $a_1 = 2$

$3a_1 - a_2 + 3a_3 = 0$ $a_2 = 3$

$a_1 + 2a_2 + 8a_3 = 0$ $a_3 = -1$

So at least one scalar is non-zero, a_1, a_2, a_3

So $(2,3,1), (1,-1,2), (7,3,8)$ is linearly dependent in $\mathbb{F}^3$.

- Every list of vectors in V containing the zero vector, such as

$$v_1, v_2, 0, v_4, v_5, v_6,$$

is linearly dependent.

Suppose $a_1, a_2, a_3, a_4, a_5, a_6 \in \mathbb{F}$ satisfy

$$a_1 v_1 + a_2 v_2 + a_3 (0) + a_4 v_4 + a_5 v_5 + a_6 v_6 = 0$$

Then set

$$a_1 = 0, a_2 = 0, a_3 = 1, a_4 = 0, a_5 = 0, a_6 = 0.$$

So the scalars are NOT ALL ZERO.

7/2

2.21 Linear Dependence Lemma

Suppose $v_1, \dots, v_m$ is a linearly dependent list in V.
Then there exists $j \in \{1, 2, \dots, m\}$ such that:

(a) $v_j \in \mathrm{span}(v_1, \dots, v_{j-1})$ (linear combination of vectors in span)
($v_j = a_1 v_1 + \dots + a_{j-1} v_{j-1}$ for some $a_1, \dots, a_{j-1} \in \mathbb{F}$)

(b) If the jth term is removed from $v_1, \dots, v_m$,
(resulting in $v_1, \dots, v_{j-1}, v_{j+1}, \dots, v_m$)
then $\mathrm{span}(v_1, \dots, v_{j-1}, v_{j+1}, \dots, v_m) = \mathrm{span}(v_1, \dots, v_m)$.

Proof of (a):

Since $v_1, \dots, v_m$ is linearly dependent, there exist $a_1, \dots, a_m \in \mathbb{F}$, not all 0, such that $a_1 v_1 + \dots + a_m v_m = 0$.

In particular, there exists $j \in \{1, \dots, m\}$ such that $a_j \neq 0$.
So we have $a_1 v_1 + \dots + a_{j-1} v_{j-1} + a_j v_j + \dots + a_m v_m = 0$

Solve for v_j:
$$v_j = -\frac{a_1}{a_j} v_1 - \dots - \frac{a_{j-1}}{a_j} \cdot v_{j-1}$$

In other words, v_j is a linear combination of $v_1, \dots, v_{j-1}$.
Therefore, $v_j \in \mathrm{span}(v_1, \dots, v_{j-1})$, proving (a).

Proof of (b):

Suppose $u \in \text{span}(v_1, .., v_m)$. Then there exist $c_1, .., c_m \in \mathbb{F}$ such that

$$u = c_1 v_1 + .. + c_m v_m.$$

Recall from the proof of part (a)

$$v_j = -\frac{a_1}{a_j} v_1 - .. - \frac{a_{j-1}}{a_j} v_{j-1}$$

We have

$$u = c_1 v_1 + .. + c_m v_m$$
$$= c_1 v_1 + .. + c_{j-1} v_{j-1} + c_j v_j + c_{j+1} v_{j+1} + .. + c_m v_m$$
$$= c_1 v_1 + .. + c_{j-1} v_{j-1} + c_j \left(-\frac{a_1}{a_j} v_1 - .. - \frac{a_{j-1}}{a_j} v_{j-1}\right) + c_{j+1} v_{j+1} + .. + c_m v_m$$
$$= \left(c_1 - c_j \frac{a_1}{a_j}\right) v_1 + .. + \left(c_j + c_j \frac{a_{j-1}}{a_j}\right) v_{j-1} + c_{j+1} v_{j+1} + .. + c_m v_m$$
$$\in \text{span}(v_1, .., v_{j-1}, v_{j+1}, .., v_m)$$

Therefore, $\text{span}(v_1, .., v_m) \subset \text{span}(v_1, .., v_{j-1}, v_{j+1}, .. v_m)$.

On the other hand, let $u \in \text{span}(v_1, .. v_{j-1}, v_{j+1}, .., v_m)$. Then there exist $a_1, .., a_m \in \mathbb{F}$ such that

$$u = a_1 v_1 + .. a_{j-1} v_{j-1} + a_{j+1} v_{j+1} + .. + a_m v_m$$

But also,

$$u = a_1 v_1 + .. + a_{j-1} v_{j-1} + \boxed{0 v_j} + a_{j+1} v_{j+1} + .. + a_m v_m$$
$$\in \text{span}(v_1, .., v_m)$$

So $\text{span}(v_1, .., v_{j-1}, v_{j+1}, .., v_m) \subset \text{span}(v_1, .., v_m)$

Therefore, we have the set equality

$$\text{span}(v_1, .., v_{j-1}, v_{j+1}, .., v_m) = \text{span}(v_1, .., v_m),$$

proving part (b).

2.24 Example | Is the list $(1,2,3),(4,5,8),(9,6,7),$ $(-3,2,8)$ linearly independent in $\mathbb{R}^3$?

No.
The list $(1,0,0),(0,1,0),(0,0,1)$ spans $\mathbb{R}^3$.
This list has length 3.

- No list of length greater than 3 ~~spans $\mathbb{R}^3$~~ is linearly independent in $\mathbb{R}^3$. Because ~~the fourth vector~~ any vector in the list $(1,2,3),(4,5,8),(9,6,7),(-3,2,8)$ is a linear combination of the other three.

2.25 Example | Does the list $(1,2,3,-5),(4,5,8,3),$ $(9,6,7,-1)$ span $\mathbb{R}^4$?

No. The list $(1,0,0,0),(0,1,0,0),(0,0,1,0),(0,0,0,1)$ spans $\mathbb{R}^4$.
No list of length less than 4 spans $\mathbb{R}^4$.

2.26 Finite dimensional subspaces

Every subspace of a finite-dimensional vector space is finite dimensional.

(We're skipping the proof in our lecture)