/11/2019

2A: Span and Linear Independence

possible problems from HW 2 on Group Exam 2

- SUBSET of your HW 2
  - 2A ~ 1, 6, 7, 8, 9, 10
  - 2B ~ 6, 8
  - 2C ~ 16

2.3 **Defn**
A linear combination of a list $v_1, \cdots, v_m \in V$ is a vector of the form

$$a_1 v_1 + \cdots + a_m v_m$$

for some $a_1, \cdots, a_m \in \mathbb{F}$.

Eg Let $V = \mathbb{F}^3$. Is $(17, -4, 2)$ a linear combination of the list $(2, 1, -3), (1, -2, 4)$?

If yes, then there exist $a_1, a_2 \in \mathbb{F}$ such that

$$(17, -4, \overset{5}{2}) = a_1(2, 1, -3) + a_2(1, -2, 4)$$

That is, we have

$$\begin{aligned}(17, -4, \overset{5}{2}) &= (2a_1, a_1, -3a_1) + (a_2, -2a_2, 4a_2)\\ &= (2a_1 + a_2, a_1 - 2a_2, -3a_1 + 4a_2)\end{aligned}$$

$$\left.\begin{aligned} 17 &= 2a_1 + a_2 \\ -4 &= a_1 - 2a_2 \end{aligned}\right\} \Rightarrow a_1 = 6, \; a_2 = 5$$

(can just use any 2 to solve for a_1 and a_2)

$$\underset{5}{2} \overset{?}{=} -3a_1 + 4a_2 \Rightarrow 2 \overset{\checkmark}{=} -3(6) + 4(5)$$

$$5 \neq -3(6) + 4(5)$$

Therefore, $(17, -4, 2)$ is a linear combination of $(2, 1, -3), (1, -2, 4)$ but $(17, -4, 5)$ is NOT a linear combination of $(2, 1, -3), (1, -2, 4)$.

2.5 **Defn** The span of $v_1, \cdots, v_m$ is the set of all linear combinations of $v_1, \cdots, v_m \in V$, and is denoted

$$\operatorname{span}(v_1, \cdots, v_m) = \{\underbrace{a_1 v_1 + \cdots + a_m v_m}_{\text{linear combination of } v_1, \cdots, v_m} : a_1, \cdots, a_m \in \mathbb{F}\}$$

Note →

(The span of the empty list ( ) is defined to be $\{0\}$.)

2.6 Eg In $V = \mathbb{F}^3$, • $(17, -4, 2)$ is a linear combination of $(2, 1, -3), (1, -2, 4)$

* Therefore, $(17, -4, 2) \in \text{span}((2, 1, -3), (1, -2, 4))$

• $(17, -4, 5)$ is NOT a linear combination of $(2, 1, -3), (1, -2, 4)$. Therefore, $(17, -4, 5) \in \text{span}((2, 1, -3), (1, -2, 4))$

Thm 2.7 <u>Span is the smallest containing subspace</u>

The span of a list of vectors $v_1, \cdots, v_m \in V$ is the smallest subspace of V containing $v_1, \cdots, v_m$.

<u>proof</u>: First[1], we will prove that $\text{span}(v_1, \cdots, v_m)$ is a subspace of V.

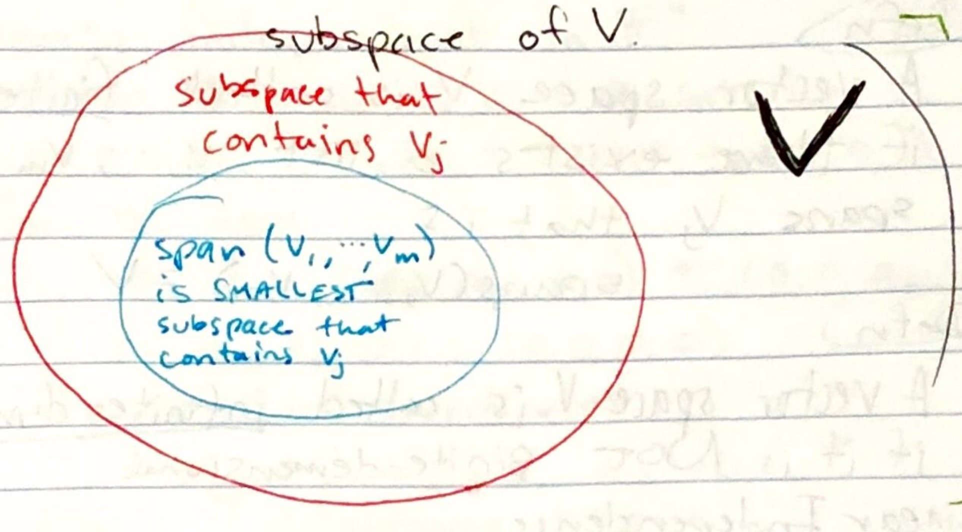
- Additive Identity: $0 = \underline{0}\, v_1 + \cdots + \underline{0}\, v_m \in \text{span}(v_1, \cdots, v_m)$
- Closed Under Addition: Let $a_1 v_1 + \cdots + a_m v_m, c_1 v_1 + \cdots c_m v_m \in \text{span}(v_1, \cdots, v_m)$ for some $a_1, \cdots, a_m, c_1, \cdots, c_m \in \mathbb{F}$ we have $(a_1 v_1 + \cdots + a_m v_m) + (c_1 v_1 + \cdots + c_m v_m)$ $= (a_1 + c_1) v_1 + \cdots + (a_m + c_m) v_m \in \text{span}(v_1, \cdots, v_m)$
- Closed Under Scalar Multiplication:
  Let $\lambda \in \mathbb{F}$ be arbitrary. Then
  $\lambda(a_1 v_1 + \cdots + a_m v_m) = (\lambda a_1) v_1 + \cdots + (\lambda a_m) v_m \in \text{span}(v_1, \cdots, v_m)$

Therefore, $\text{span}(v_1, \cdots, v_m)$ is a subspace of V.

- Now[2], we will prove that $\text{span}(v_1, \cdots, v_m)$ is the SMALLEST subspace of V.
- First, note that each v_j $(j = 1, \cdots, m)$ can be
  $v_j = 0 v_1 + \cdots + 0 v_{j-1} + 1 v_j + 0 v_{j+1} + \cdots + 0 v_m \in \text{span}(v_1, \cdots, v_m)$
  - In other words, $\text{span}(v_1, \cdots, v_m)$ contains each v_j, or equivalently $\text{span}(v_1, \cdots, v_m)$ contains $v_1, \cdots, v_m$

- Because every subspace of V is closed under scalar multiplication and addition, every subspace containing v_j contains all linear combination of $v_1, \dots, v_m$.
  - In other words, every subspace contains $\text{span}(v_1, \dots, v_m)$.
  - This makes $\text{span}(v_1, \dots, v_m)$ the SMALLEST subspace of V.

Visual:



2.8 Defn

If we have $\text{span}(v_1, \dots, v_m) = V_j$, then we say $v_1, \dots, v_m$ spans V.

2.9 Eg (modifed) The list $(1,0,0), (0,1,0), (0,0,1)$ spans $\mathbb{F}^3$

proof: Let $(x_1, x_2, x_3) \in \mathbb{F}^3$ be arbitrary.

Then we can write

$$(x_1, x_2, x_3) = (x_1, 0, 0) + (0, x_2, 0) + (0, 0, x_3)$$
$$= x_1(1,0,0) + x_2(0,1,0) + x_3(0,0,1)$$

so, ~~(1,0,0)~~ $(x_1, x_2, x_3) \in \text{span}((1,0,0), (0,1,0), (0,0,1))$.

In other words, $(1,0,0), (0,1,0), (0,0,1)$ spans $\mathbb{F}^3$

2.11 Defn. • a function $p: \mathbb{F} \to \mathbb{F}$ is a polynomial if there exists $a_0, \dots, a_m \in \mathbb{F}$ such that

$$p(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \dots + a_m z^m$$

for all $z \in \mathbb{F}$

- The set of all polynomials with coefficients in $\mathbb{F}$ is called $\mathcal{P}(\mathbb{F})$

2.12 Defn A polynomial $p \in \mathcal{P}(\mathbb{F})$ is said to have degree m if there exist $a_0, a_1, a_2, \cdots, a_m \in \mathbb{F}$ with $a_m \neq 0$ such that

~~2.10 Defn~~

$$p(z) = a_0 + a_1 z + \cdots + a_m z^m$$

for all $z \in \mathbb{F}$

2.10 Defn

A vector space V is called finite-dimensional if there exists a list $v_1, \cdots, v_m \in V$ that spans V; that is

$$\text{spans}(v_1, \cdots, v_m) = V$$

2.15 Defn

A vector space V is called infinite-dimensional if it is NOT finite-demensional.

Linear Independence

2.17 Defn

A list $v_1, \cdots, v_m \in V$ is called linearly Independent if the only choice of $a_1, \cdots, a_m \in \mathbb{F}$ that satisfies

$$a_1 v_1 + \cdots + a_m v_m = 0$$

is $a_1 = 0, \cdots, a_m = 0$.

2.19 Defn

A list $v_1, \cdots, v_m \in V$ is linearly Dependent if there exist $a_1, \cdots, a_m \in \mathbb{F}$, not all zero (in other words, some nonzero) such that

$$a_1 v_1 + \cdots a_m v_m = 0.$$

In other words, $v_1, \cdots, v_m \in V$ is NOT linearly independent.

2.18 **Eg** LINEARLY INDEPENDENT LISTS

(a) A list v_1 of one vector $v_1 \in V$ is linearly independent if and only if $v_1 \neq 0$. Then if $a_1 \in \mathbb{F}$ satisfies

$$a_1 v_1 = 0,$$

then $$a_1 = 0.$$

(b) A list $(1,0,0,0), (0,1,0,0), (0,0,1,0)$ is linearly independent in $\mathbb{F}^4$.

Suppose $a_1, a_2, a_3, a_4 \in \mathbb{F}$ satisfy

$$a_1(1,0,0,0) + a_2(0,1,0,0) + a_3(0,0,1,0) = 0$$

Then we have

$$(a_1,0,0,0) + (0,a_2,0,0) + (0,0,a_3,0) = (0,0,0,0)$$

$$\Rightarrow (a_1,a_2,a_3,0) = (0,0,0,0)$$

Equate the coordinates

$$a_1 = 0, \quad a_2 = 0, \quad a_3 = 0.$$ ✓

2.20 **Eg** LINEARLY DEPENDANT LISTS

(a) $(2,3,1), (1,-1,2), (7,3,8)$ is linearly dependent in $\mathbb{F}^3$

Suppose $a_1, a_2, a_3 \in \mathbb{F}$ satisfy

$$a_1(2,3,1) + a_2(1,-1,2) + a_3(7,3,8) = (0,0,0)$$

$$(2a_1, 3a_1, a_1) + (a_2, -a_2, 2a_2) + (7a_3, 3a_3, 8a_3) = (0,0,0)$$

$$(2a_1 + a_2 + 7a_3) + (3a_1 - a_2 + 3a_3) + (a_1 + 2a_2 + 8a_3) = (0,0,0)$$

$$\left.\begin{aligned} &2a_1 + a_2 + 7a_3 \\ &3a_1 - a_2 + 3a_3 \\ &a_1 + 2a_2 + 8a_3 \end{aligned}\right\} \xrightarrow[\text{solved}]{\text{system}} \boxed{\begin{aligned} a_1 &= 2 \\ a_2 &= 3 \\ a_3 &= -1 \end{aligned}}$$

So, at least one scalar is non-zero.

$$a_1, a_2, a_3$$

So, $(2,3,1), (1,-1,2), (7,3,8)$ is linearly dependent in $\mathbb{F}^3$.

(b) Every list of vectors in V containing the zero vector, such as

$$v_1, v_2, 0, v_4, v_5, v_6$$

is linearly dependent.

Suppose $a_1, a_2, a_3, a_4, a_5, a_6 \in \mathbb{F}$ satisfy

$$a_1 v_1 + a_2 v_2 + a_3 (0) + a_4 v_4 + a_5 v_5 + a_6 v_6 = 0$$

Then set $a_1 = 0,\ a_2 = 0,\ a_3 = 1,\ a_4 = 0,\ a_5 = 0,\ a_6 = 0$

So, the scalars are NOT ALL zero

7/2/2019 2A Lets Revisit:

2.7 Defn: Span is the smallest containing subspace

Every subspace of V containing $v_1, \cdots, v_m$ (each v_j, $j=1,\cdots,m$)
contains $\operatorname{span}(v_1, \cdots, v_m)$

In other words:

If a subspace contains $v_1, \cdots, v_m$, then it contains $\operatorname{span}(v_1, \cdots, v_m)$

Proof of that statement:

Let U be a subspace of V that contains $v_1, \cdots, v_m$. Since subspaces are closed under addition and scalar multiplication, $v_1, \cdots, v_m \in U$ implies

$$a_1 v_1 + \cdots + a_m v_m \in U$$

for all $a_1, \cdots, a_m \in \mathbb{F}$. By definition,

$$\operatorname{span}(v_1, \cdots, v_m) = \{ a_1 v_1 + \cdots + a_m : a_1, \cdots, a_m \in \mathbb{F} \},$$

Therefore, $\operatorname{span}(v_1, \cdots, v_m) \subset U$.

In other words, every subspace of V that contains $v_1, \cdots, v_m \in V$ must also contain $\operatorname{span}(v_1, \cdots, v_m)$. This makes $\operatorname{span}(v_1, \cdots, v_m)$ the SMALLEST subspace of V that contains $v_1, \cdots, v_m$.

2.21 Linear Dependence Lemma

Suppose $v_1, \cdots, v_m$ is a linearly dependent list in V. then there exist $j \in \{1, 2, \cdots, m\}$ such that:

(a) $v_j \in \operatorname{span}(v_1, \cdots, v_{j-1})$
($v_j = a_1 v_1 + \cdots + a_{j-1} v_{j-1}$ for some $a_1, \cdots, a_{j-1} \in \mathbb{F}$)

(b) if the j^{th} term is removed from $v_1, \cdots, v_m$ (resulting in $v_1, \cdots, v_{j-1}, v_{j+1}, \cdots, v_m$)
then $\operatorname{span}(v_1, \cdots, v_{j-1}, v_{j+1}, \cdots, v_m) = \operatorname{span}(v_1, \cdots, v_m)$

proof of (a):

since $v_1, \cdots, v_m$ is linearly dependent, there →

exist $a_1, \cdots, a_m \in \mathbb{F}$, not all 0, such that

$$a_1 v_1 + \cdots + a_m v_m = 0.$$

In particular, there exists $j \in \{1, \cdots, m\}$ such that $a_j \neq 0$.

So we have:

$$a_1 v_1 + \cdots + a_{j-1} v_{j-1} + a_j v_j + \cdots + a_m v_m = 0.$$

Solve for v_j:

$$v_j = -\frac{a_1}{a_j} v_1 - \cdots - \frac{a_{j-1}}{a_j} v_{j-1}.$$

In other words, v_j is a linear combination of $v_1, \cdots, v_{j-1}$.

Therefore, $v_j \in \operatorname{span}(v_1, \cdots, v_{j-1})$, proving (a).

Proof of (b):

Suppose $u \in \operatorname{span}(v_1, \cdots, v_m)$. There exist $c_1, \cdots, c_m \in \mathbb{F}$ such that $u = c_1 v_1 + \cdots + c_m v_m$.

Recall from the proof of part (a):

$$v_j = -\frac{a_1}{a_j} v_1 - \cdots - \frac{a_{j-1}}{a_j} v_{j-1}.$$

We have:

$$\begin{aligned} u &= c_1 v_1 + \cdots + c_m v_m \\ &= c_1 v_1 + \cdots + c_{j-1} v_{j-1} + c_j v_j + c_{j+1} v_{j+1} + \cdots c_m v_m \\ &= \boxed{c_1 v_1} + \cdots + \boxed{c_{j-1} v_{j-1}} + \boxed{c_j}\left(\boxed{-\frac{a_1}{a_j} v_1} - \cdots \boxed{-\frac{a_{j-1}}{a_j} v_{j-1}}\right) + c_{j+1} v_{j+1} + \cdots + c_m v_m \\ &= \left(c_1 - c_j \frac{a_1}{a_j}\right) v_1 + \cdots + \left(c_{j-1} - c_j \frac{a_{j-1}}{a_j}\right) v_{j-1} + c_{j+1} v_{j+1} + \cdots + c_m v_m \\ &\in \operatorname{span}(v_1, \cdots, v_{j-1}, v_{j+1}, \cdots, v_m). \end{aligned}$$

Therefore, $\operatorname{span}(v_1, \cdots, v_m) \subset \operatorname{span}(v_1, \cdots, v_{j-1}, v_{j+1}, \cdots, v_m)$.

On the other hand, let $u \in \operatorname{span}(v_1, \cdots, v_{j-1}, v_{j+1}, \cdots, v_m)$

Then there exist $a_1, \cdots, a_m \in \mathbb{F}$ such that

$$u = a_1 v_1 + \cdots + a_{j-1} v_{j-1} + a_{j+1} v_{j+1} + \cdots + a_m v_m$$

But also,

$$u = a_1 v_1 + \cdots + a_{j-1} v_{j-1} + \boxed{0 v_j} + a_{j+1} v_{j+1} + \cdots + a_m v_m$$

$\operatorname{span} \in (v_1, \cdots, v_m)$

So $\operatorname{span}(v_1, \dots, v_{j-1}, v_{j+1}, \dots v_m) \subset \operatorname{span}(v_1, \dots, v_m)$.

Therefore, we have the set equality

$\operatorname{span}(v_1, \dots, v_{j-1}, v_{j+1}, \dots, v_m) = \operatorname{span}(v_1, \dots, v_m)$.

proving part (b). $\square$

2.24 Eg Is the list $(1,2,3), (4,5,8), (2,6,7), (-3,2,8)$ linearly independent in $\mathbb{R}^3$?

The list $(1,0,0), (0,1,0), (0,0,1)$ spans $\mathbb{R}^3$.

This list has length 3.

No list of length 3 is linearly independent in $\mathbb{R}^3$

Because any vector in the list

$(1,2,3), (4,5,8), (2,6,7), (-3,2,8)$

is a linear combination of the other three.

2.25 Eg Does the list $(1,2,3,-4), (4,5,8,3), (2,6,7,-1)$ span $\mathbb{R}^4$?

No. the list $(1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1)$ spans $\mathbb{R}^4$.

No list of length less than 4 spans $\mathbb{R}^4$.

2.28 Defn <u>Finite dimensional subspaces</u>

Every subspace of a finite-dimensional vector space is finite dimensional.

we're skipping the proof, see textbook.