2.A Span and Linear Independence

2.3 Definition: A linear combination of a list $v_1, \ldots, v_m \in V$ is a vector of the form

$a_1 v_1 + \ldots + a_m v_m$

for some $a_1, \ldots, a_m \in \mathbb{F}$

Example:

Let $V = \mathbb{F}^3$. Is $(17, -4, 2)$ a linear combination of the list $(2, 1, -3)$, $(1, -2, 4)$?

If yes, then there exist $a_1, a_2 \in \mathbb{F}$ such that

$(17, -4, 2) = a_1 (2, 1, -3) + a_2 (1, -2, 4)$

That is, we have

$(17, -4, 2) = (2a_1, a_1, -3a_1) + (a_2, -2a_2, 4a_2)$

$= (2a_1 + a_2, a_1 - 2a_2, -3a_1 + 4a_2)$

$\left.\begin{aligned} 17 &= 2a_1 + a_2 \\ -4 &= a_1 - 2a_2 \end{aligned}\right\}$ we chose these two to solve for unknowns

$2 = -3a_1 + 4a_2$ $\Rightarrow a_1 = 6,\ a_2 = 5$

What about $(17, -4, 5)$?

check by third eqtn

$2 = -3(6) + 4(5)$ ✓

Therefore, $(17, -4, 2)$ is a linear combination of $(2, 1, -3)$, $(1, -2, 4)$, but $(17, -4, 5)$ is NOT a linear combo of $(2, 1, -3)$, $(1, -2, 4)$.

2.5 Definition: The span of $v_1, \ldots, v_m$ is the set of all linear combinations of $v_1, \ldots, v_m \in V$, and is denoted

$\text{span}(v_1, \ldots, v_m) = \{a_1 v_1 + \ldots + a_m v_m \in V : a_1, \ldots, a_m \in \mathbb{F}\}$

↑ linear combo of $v_1, \ldots, v_m$

(The span of the empty list () is defined to be $\{0\}$.)

2.6 Example In $V = \mathbb{F}^3$, • $(17, -4, 2)$ is a lin combo of $(2, 1, -3)$, $(1, -2, 4)$

Therefore, $(17, -4, 2) \in \text{span}((2, 1, -3), (1, -2, 4))$

• $(17, -4, 5)$ is NOT a lin combo of $(2, 1, -3)$, $(1, -2, 4)$

Therefore, $(17, -4, 5) \notin \text{span}((2, 1, -3), (1, -2, 4))$

2.7 Span is the smallest continuing subspace

The span of a list of vectors $v_1, \ldots, v_m \in V$ is the smalle subspace of V continuing $v_1, \ldots, v_m$.

Proof: First, we will prove that $\text{span}(v_1, \ldots, v_m)$ is a subspace of V

- Additive identity: $0 = 0v_1 + \ldots + 0v_m \in \text{span}(v_1, \ldots, v_m)$

- Closed under Addition: Let $a_1v_1 + \ldots + a_mv_m$, $c_1v_1 + \ldots + c_m$ $\in \text{span}(v_1, \ldots, v_m)$, for some $a_1, \ldots, a_m, c_1, \ldots, c_m \in \mathbb{F}$. Th we have $(a_1v_1 + \ldots + a_mv_m) + (c_1v_1 + \ldots + c_mv_m)$
  $= (a_1 + c_1)v_1 + \ldots + (a_m + c_m)v_m \in \text{span}(v_1, \ldots, v_m)$

- Closed Under Scalar Multiplication: Let $\lambda \in \mathbb{F}$ be arbitrar Then $\lambda(a_1v_1 + \ldots + a_mv_m) = (\lambda a_1)v_1 + \ldots + (\lambda a_m)v_m \in \text{span}$ $v_1, \ldots, v_m)$

Therefore, $\text{span}(v_1, \ldots, v_m)$ is a subspace of V.

Now we will prove that $\text{span}(v_1, \ldots, v_m)$ is the SMALLE subspace of V

First, notice that each $v_j =$ for all $(j = 1, \ldots, m)$ can be written as a linear combo of $v_1, \ldots, v_m$:

$v_j = 0v_1 + \ldots + 0v_{j-1} + 1v_j + 0v_{j+1} + \ldots + 0v_m \in \text{span}(v_1, \ldots, v_m$

In other words, $\text{span}(v_1, \ldots, v_m)$ contains each v_j, or equivalently $\text{span}(v_1, \ldots, v_m)$ contains $v_1, \ldots, v_m$. Also b/c every subspace of V is closed under scalar mult. & addition, every subspace containing v_j contains <u>all</u> linear combo of $v_1, \ldots, v_m$. In other words every subspace contains $\text{span}(v_1, \ldots, v_m)$. This makes $\text{span}(v_1, \ldots, v_m)$ the SMALLEST subspace of V.

2.8 Definition:

If we have $\text{span}(v_1, \ldots, v_m) = V$, then we say $v_1, \ldots, v_m$ <u>spans</u> V.

2.9 Example:

The list $(1,0,0), (0,1,0), (0,0,1)$ spans $\mathbb{F}^3$.

Proof: Let $(x_1, x_2, x_3) \in \mathbb{F}^3$ be arbitrary... Then we write

$$(x_1, x_2, x_3) = (x_1, 0, 0) + (0, x_2, 0) + (0, 0, x_3)$$
$$= x_1(1,0,0) + x_2(0,1,0) + x_3(0,0,1)$$

So $(x_1, x_2, x_3) \in \text{span}((1,0,0), (0,1,0), (0,0,1))$. In other words, $(1,0,0), (0,1,0), (0,0,1)$ spans $\mathbb{F}^3$.

2.11 Definition

- A function $p: \mathbb{F} \to \mathbb{F}$ is a polynomial if there exist $a_0, \dots, \in \mathbb{F}$ such that $p(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \dots + a_m z^m$ for all $z \in \mathbb{F}$
- The set of all polynomials w/ coefficients in $\mathbb{F}$ is called $P(\mathbb{F})$.

2.12 A Polynomial $p \in P(\mathbb{F})$ is said to be degree m if there exist $a_0, a_1, \dots, a_m \in \mathbb{F}$ w/ $a_m \neq 0$ such that $P(z) = a_0 + a_1 z + a_m z^m$ for all $z \in \mathbb{F}$

2.10 Definition: A vector space V is called finite-dimensional if there exists a list $v_1, \dots, v_m \in V$ that spans V; that is $\text{span}(v_1, \dots, v_m) = V$.

2.15 Definition: A vector space V is called infinite-dimensional if it is NOT finite dimensional.

Linear Independence

2.17 Definition

- A list $v_1, \dots, v_m \in V$ is called linear independent if the only choice of $a_1, \dots, a_m \in \mathbb{F}$ that satisfies $a_1 v_1 + \dots + a_m v_m = 0$ is $a_1 = 0, \dots, a_m = 0$.

2.19 Definition

- A list $v_1, \dots, v_m \in V$ is linearly dependent if there exist $a_1, \dots, a_m \in \mathbb{F}$ not all zero (in other words, some nonzero) such that $a_1 v_1 + \dots + a_m v_0 = 0$

In other words, $v_1, \dots, v_m \in V$ is NOT linearly independent

2.18 Example Linearly Independent lists

- A list v_1 of one vector $v \in V$ is linearly independent if is only $v_1 \neq 0$. Then if $a_1 \in \mathbb{F}$ satisfies $a_1 v_1 = 0$, then $a_1 = 0$.
- A list $(1,0,0,0), (0,1,0,0), (0,0,1,0)$ is linearly independent in $\mathbb{F}^4$.

Suppose $a_1, a_2, a_3, a_4 \in \mathbb{F}$ satisfy, $a_1(1,0,0,0) + a_2(0,1,0,0) + a_3(0,0,1,0) = (0,0,0,0)$

Then we have $(a_1,0,0,0) + (0,a_2,0,0) + (0,0,a_3,0) = (0,0,0,0)$

$(a_1, a_2, a_3, 0) = (0,0,0,0)$

Equate the coordinates

$a_1 = 0, a_2 = 0, a_3 = 0$

2.20 Example Linearly Dependent lists

- $(2,3,1), (1,-1,2), (7,3,8)$ is linearly dependent in $\mathbb{F}^3$

suppose $a_1, a_2, a_3 \in \mathbb{F}$ satisfy

$a_1(2,3,1) + a_2(1,-1,2) + a_3(7,3,8) = (0,0,0)$

$(2a_1, 3a_1, a_1) + (a_2, -a_2, 2a_2) + (7a_3, 3a_3, 8a_3) = (0,0,0)$

$(2a_1 + a_2 + 7a_3,\ 3a_1 - a_2 + 3a_3,\ a_1 + 2a_2 + 8a_3) = (0,0,0)$

$2a_1 + a_2 + 7a_3 = 0$ $\qquad a_1 = 2$

$3a_1 - a_2 + 3a_3 = 0 \implies a_2 = 3$

$a_1 + 2a_2 + 8a_3 = 0$ $\qquad a_3 = -1$

so at least one scalar is nonzero a_1, a_2, a_3

so $(2,3,1), (1,-1,2), (7,3,8)$ is linearly dependent in $\mathbb{F}^3$

- Every list of vectors in V containing the zero vector, such as $v_1, v_2, 0, v_4, v_5, v_6$ is linearly dependent

suppose $a_1, a_2, a_3, a_4, a_5, a_6 \in \mathbb{F}$ satisfy

$a_1v_1 + a_2v_2 + a_3(0) + a_4v_4 + a_5v_5 + a_6v_6 = 0$

Then set

$a_1 = 0, a_2 = 0, a_3 = 1, a_4 = 0, a_5 = 0, a_6 = 0$

so the scalars are NOT ALL ZERO.

7/2/19 2.A
week 2
Tuesday

Lets revisit:

2.7 Span is the smallest containing subspace

Every subspace of V containing $v_1, \dots, v_m$ each v_j $j=1,\dots,m$ contains $\text{span}(v_1, \dots, v_m)$

In other words: If a subspace contains $v_1, \dots, v_m$, then it contains $\text{span}(v_1, \dots, v_m)$

Proof of that statement:

Let U be a subspace of V that contains $v_1, \dots, v_m$. Since subspaces are closed under addition & scalar mult., $v_1, \dots, v_m \in U$ implies $a_1v_1 + \dots + a_mv_m \in U$ for all $a_1, \dots, a_m \in \mathbb{F}$. By definition, $\text{span}(v_1, \dots, v_m) = \{a_1v_1 + \dots + a_m : a_1, \dots, a_m \in \mathbb{F}\}$.

Therefore, $\text{span}(v_1, \dots, v_m) \in U$

In other words, every subspace of V that contains $v_1, \dots, v_m \in V$ must also contain $\text{span}(v_1, \dots, v_m)$ the SMALLEST subspace of V that contains $v_1, \dots, v_m$

2.21 Linear Dependence Lemma

Suppose $v_1, \ldots, v_m$ is a linearly dependent list in V. Then there ex
$j \in \{1, 2, \ldots, m\}$ such that:

(a) $v_j \in \text{span}(v_1, \ldots, v_{j-1})$
linear combination

(b) If the j^{th} term is removed from $v_1, \ldots, v_m$, (resulting in $v_1, \ldots, v_{j-1}$, $v_{j+1}, \ldots, v_m$)
then $\text{span}(v_1, \ldots, v_{j-1}, v_{j+1}, \ldots, v_m) = \text{span}(v_1, \ldots, v_m)$

Proof of (a): Since $v_1, \ldots, v_m$ is linearly dependent, there exis $a_1, \ldots, a_m \in \mathbb{F}$, not all 0, such that $a_1v_1 + \ldots + a_mv_m = 0$.
In particular, there exists $j \in \{1, \ldots, m\}$ such that $a_j \neq 0$. So we have $a_1v_1 + \ldots + a_{j-1}v_{j-1} + a_jv_j + \ldots + a_mv_m = 0$.
Solve for v_j:

$$v_j = \frac{-a_1}{a_j} v_1 - \ldots - \frac{a_{j-1}}{a_j} v_{j-1}$$

In other words, v_j is a linear combination of $v_1, \ldots, v_{j-1}$.
Therefore, $v_j \in \text{span}(v_1, \ldots, v_{j-1})$, proving (a).

Proof of (b):
Suppose $u \in \text{span}(v_1, \ldots, v_m)$. Then there exist $c_1, \ldots, c_m \in \mathbb{F}$ such that $u = c_1v_1 + \ldots + c_mv_m$.
Recall from the proof of part (a): $v_j = \frac{-a_1}{a_j} v_1 - \ldots - \frac{a_{j-1}}{a_j} v_{j-1}$
we have

$$\begin{aligned} u &= c_1v_1 + \ldots + c_mv_m \\ &= c_1v_1 + \ldots + c_{j-1}v_{j-1} + c_jv_j + c_{j+1}v_{j+1} + \ldots + c_mv_m \\ &= c_1v_1 + \ldots + c_{j-1}v_{j-1} + c_j\left(\frac{-a_1}{a_j} v_1 - \ldots - \frac{a_{j-1}}{a_j} v_{j-1}\right) + c_{j+1}v_{j+1} + \ldots + c_mv_m \\ &= \left(c_1 - c_j\frac{a_1}{a_j}\right)v_1 + \ldots + \left(c_{j-1} - c_j\frac{a_{j-1}}{a_j}\right)v_{j-1} + c_{j+1}v_{j+1} + \ldots + c_mv_m \end{aligned}$$

$\in \text{span}(v_1, \ldots, v_{j-1}, v_{j+1}, v_m)$
Therefore, $\text{span}(v_1, \ldots, v_m) \subset \text{span}(v_1, \ldots, v_{j-1}, v_{j+1}, \ldots, v_m)$
On the other hand, let $u \in \text{span}(v_1, v_{j-1}, v_{j+1}, \ldots, v_m)$
Then there exist $a_1, \ldots, a_m \in \mathbb{F}$ such that
$u = a_1v_1 + \ldots + a_{j-1}v_{j-1} + a_{j+1}v_{j+1} + \ldots + a_mv_m$.
But also,
$u = a_1v_1 + \ldots + a_{j-1}v_{j-1} + 0v_j + a_{j+1}v_{j+1} + \ldots + a_mv_m \in \text{span}(v_1, \ldots$
So $\text{span}(v_1, \ldots, v_{j-1}, v_{j+1}, \ldots, v_m) \subset \text{span}(v_1, \ldots, v_m)$
Therefore, we have the set equality $\text{span}(v_1, \ldots, v_{j-1}, v_{j+1}, \ldots, v_m) = \text{span}(v_1, \ldots, v_m) \Rightarrow$ Proving part (b).

2.24 Example

Is the list $(1,2,3), (4,5,8), (9,6,7), (-3,2,8)$ linearly independent in $\mathbb{R}^3$? NO

The list $(1,0,0), (0,1,0), (0,0,1)$ spans $\mathbb{R}^3$. This list has length 3

- No list of length greater than 3 is lin. indep. in $\mathbb{R}^3$, b/c any vector in the list $(1,2,3), (4,5,8), (9,6,7), (-3,2,8)$ is a linear combo of the other three

2.25 Example Does the list $(1,2,3,-5), (4,5,8,3), (9,6,7,-1)$ span $\mathbb{R}^4$?

No, the list $(1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1)$ spans

No list of length less than 4 spans $\mathbb{R}^4$.

2.26 Finite Dimensional Subspaces

Every subspace of a finite dimensional vector space is finite dimensional. (skipping the proof in lecture)

2.B Bases

2.27 Definition: A basis of V is a list of vectors $v_1, \ldots, v_m$ of V that is linearly independent & spans V.

2.28 Example

- The list $(1,0,\ldots,0), (0,1,0,\ldots,0), \ldots, (0,\ldots,0,1)$ is a basis of $\mathbb{F}^n$. It is called the standard basis of $\mathbb{F}^n$
- $(1,0), (0,1)$ is a basis of $\mathbb{F}^2$
- $(2,3), (4,6)$ is NOT a basis of $\mathbb{F}^2$ (is linearly dependent)
- $(1,2,3), (2,1,3)$ is NOT a basis of $\mathbb{F}^3$ (only two vectors in $\mathbb{F}^3$, therefore does not span $\mathbb{F}^3$)
- $(1,2), (3,5), (4,1,3)$ is NOT a basis of $\mathbb{F}^2$ (three vectors in $\mathbb{F}^2$, therefore one vector is linear combo of the other two) (lin. dependent).
- $(1,1,0), (0,0,1)$ is a basis of $\{(x,x,y) \in \mathbb{F}^3 : x, y \in \mathbb{F}\}$ b/c $(x,x,y) = (x,x,0) + (0,0,y)$
  $= x(1,1,0) + y(0,0,1)$
- $(1,-1,0), (1,0,-1)$ is a basis of $\mathbb{F}\{(x,y,z) \in \mathbb{F}^3 : x+y+z=0\}$ b/c $x+y+z=0$ implies $z=-x-y$.

$(x,y,z) = (x,y,-x-y)$
$= (x,0,-x) + (0,y,-y)$
$= x(1,0,-1) + y(0,1,-1)$
so $(1,0,-1), (0,1,-1)$ spans V.

Suppose $a, b \in \mathbb{F}$ satisfy
$a(1,0,-1) + b(0,1,-1) = (0,0,0)$
$(a,0,-a) + (0,b,-b) = (0,0,0)$
$(a,b,-a-b) = (0,0,0)$
Equate coordinates $a=0, b=0, -a-b=0$

Therefore the list $(1,0,-1), (0,1,-1)$ is lin. indep. So