LECTURE 05

[redacted] 07-01-19

OFFICE HOURS * New Location *

MTWR 11:10am-12:00pm @ Skye Hall [redacted] 277

Possible problems from HW 2 on Group Exam 2

$$\left.\begin{array}{ll} 2A & 1, 6, 7, 8, 9, 10 \\ 2B & 6, 8 \\ 2C & 16 \end{array}\right\} \boxed{\text{Subset of your HW 2}}$$

Section 2A Span and Linear Independence

2.3 Definition

A linear combination of a list $v_1, \ldots, v_m \in V$ is a vector of the form

$$a_1 v_1 + \cdots + a_m v_m$$

for some $a_1, \ldots, a_m \in \mathbb{F}$

Example Let $V = \mathbb{F}^3$. Is $(17, -4, 2)$ a linear combination of the list $(2, 1, -3), (1, -2, 4)$?

If yes, then there exist $a_1, a_2,$ [redacted] $\in \mathbb{F}$ such that

$$(17, -4, \underset{5}{2}) = \underline{a_1}\,(2, 1, -3) + \underline{a_2}\,(1, -2, 4)$$

That is, we have

$$(17, -4, \underset{5}{2}) = (2a_1, a_1, -3a_1) + (a_2, -2a_2, 4a_2)$$
$$= (2a_1 + a_2, a_1 - 2a_2, -3a_1 + 4a_2)$$

$$\boxed{\begin{array}{l} 17 = 2a_1 + a_2 \\ -4 = a_1 - 2a_2 \end{array}} \Rightarrow \boxed{a_1 = 6 \quad a_2 = 5}$$

$$\boxed{2 = -3a_1 + 4a_2}$$

$$\Downarrow$$

$$\boxed{2 = -3(6) + 4(5)} \checkmark$$

$$5 \neq -3(6) + 4(5)$$

Therefore, $(17, -4, 2)$ is a linear combination of $(2, 1, -3)$, $(1, -2, 4)$, but $(17, -4, 5)$ is NOT a linear combination of $(2, 1, -3)$, $(1, -2, 4)$

2.5 Definition

The span of $v_1, \dots, v_m$ is the set of all linear combinations of $v_1, \dots, v_m \in V$, and is denoted

$$\text{span}(v_1, \dots, v_m) = \{ \underbrace{a_1 v_1 + \dots + a_m v_m}_{\text{linear combination of } v_1, \dots, v_m} \in V : a_1, \dots, a_m \in \mathbb{F}$$

(The span of the empty list $()$ is defined to be $\{0\}$)

2.6 Example

In $V = \mathbb{F}^3$

- $(17, -4, 2)$ is a linear combination of $(2, 1, -3)$, $(1, -2, 4)$

  Therefore, $(17, -4, 2) \in \text{span}((2, 1, -3), (1, -2, 4))$

- $(17, -4, 5)$ is NOT a linear combination of $(2, 1, -3), (1, -2, 4)$

  Therefore, $(17, -4, 5) \notin \text{span}((2, 1, -3), (1, -2, 4))$

2.7 Span is the smallest containing subspace

The span of a list of vectors $v_1, \dots, v_m \in V$ is the smallest subspace of V containing $v_1, \dots, v_m$.

Proof: First, we will prove that $\text{span}(v_1, \dots, v_m)$ is a subspace of V

- Additive Identity: $0 = \underline{0} v_1 + \dots + \underline{0} v_m \in \text{span}(v_1, \dots, v_m)$
- Closed under addition: Let $a_1 v_1 + \dots + a_m v_m$, $c_1 v_1 + \dots + c_m v_m \in \text{span}(v_1, \dots, v_m)$, for some $a_1, \dots, a_m, c_1, \dots c_m \in \mathbb{F}$

  Then we have:

  $$(a_1 v_1 + \dots + a_m v_m) + (c_1 v_1 + \dots + c_m v_m)$$
  $$= (a_1 + c_1) v_1 + \dots + (a_m + c_m) v_m \in \text{span}(v_1, \dots, v_m)$$

- closed under scalar multiplication:

  Let $\lambda \in \mathbb{F}$ be arbitrary. Then $\lambda(a_1 v_1 + \dots + a_m v_m)$
  $= (\lambda a_1) v_1 + \dots + (\lambda a_m) v_m \in \text{span}(v_1, \dots, v_m)$

Therefore, $\text{span}(v_1, \dots v_m)$ is a subspace of V

Now we will prove that span $(v_1, \dots, v_m)$ is the smallest subspace of V.

First, notice that each v_j ($j = 1, \dots, m$) for all can be written as a linear combination of $v_1, \dots, v_m$:

$$v_j = 0v_1 + \cdots + 0_{j-1} + 1v_j + 0v_{j+1} + \cdots + 0v_m$$

$$\in \text{span}(v_1, \dots, v_m)$$

In other words, span $(v_1, \dots, v_m)$ contains each v_j, or equitably span $(v_1, \dots, v_m)$ contains $v_1, \dots, v_m$

Also because every subspace of V is closed under scalar multiplication and addition, every subspace containing v_j contains all linear combination of $v_1, \dots, v_m$. In other words, every subspace contains span $(v_1, \dots, v_m)$

This makes span $(v_1, \dots, v_m)$ the smallest subspace of V.

2.8 Definition

If we have span $(v_1, \dots, v_m) = V$, then we say $v_1, \dots, v_m$ spans V.

2.9 Example The list $(1,0,0), (0,1,0), (0,0,1)$ span $\mathbb{F}^3$

Proof: Let $(x_1, x_2, x_3) \in \mathbb{F}^3$ be arbitrary

Then we can write

$$(x_1, x_2, x_3) = (x_1, 0, 0) + (0, x_2, 0) + (0, 0, x_3)$$

$$= x_1(1,0,0) + x_2(0,1,0) + x_3(0,0,1)$$

So, $(x_1, x_2, x_3) \in$ span $((1,0,0), (0,1,0), (0,0,1))$

In other words, $(1,0,0), (0,1,0), (0,0,1)$ spans $\mathbb{F}^3$

2.11 Definition

- A function $p: \mathbb{F} \to \mathbb{F}$ is a polynomial if there exist $a_0, \dots, a_m \in \mathbb{F}$ such that

$$p(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \cdots + a_m z^m$$

for all $z \in \mathbb{F}$

- The set of all polynomials with coefficients in $\mathbb{F}$ is called $P(\mathbb{F})$

2.12 A polynomial $p \in P(\mathbb{F})$ is said to have degree m if there exist $a_0, a_1, \ldots, a_m \in \mathbb{F}$ with $a_m \neq 0$ such that

$$p(z) = a_0 + a_1 z + \cdots + a_m z^m$$

for all $z \in \mathbb{F}$

2.10 Definition

A vector space V is called finite-dimensional if there exists a list $v_1, \ldots, v_m \in V$ that spans V; that is

$$\text{span}(v_1, \ldots, v_m) = V$$

2.15 Definition

A vector space V is called infinite-dimensional if it is NOT finite dimensional

Linear Independence

2.17 Definition

- A list $v_1, \ldots, v_m \in V$ is called linearly independent if the only choice of $a_1, \ldots, a_m \in \mathbb{F}$ that satisfies

$$a_1 v_1 + \cdots + a_m v_m = 0$$

is $a_1 = 0, \ldots, a_m = 0$.

2.19 Definition

- A list $v_1, \ldots, v_m \in V$ is linearly dependent if there exist $a_1, \ldots, a_m \in \mathbb{F}$, not all zero (in other words, some nonzero) such that $a_1 v_1 + \cdots + a_m v_m = 0$
- In other words, $v_1, \ldots, v_m \in V$ is NOT linearly independent

2.18 Example LINEARLY INDEPENDENT LISTS

- A list v_1 of one vector $v_1 \in V$ is linearly independent if and only if $v \neq 0$. Then if $a_1 \in \mathbb{F}$ satisfies

$$a_1 v_1 = 0$$

then

$$a_1 = 0$$

- A list $(1,0,0,0), (0,1,0,0), (0,0,1,0)$ is linearly independent in $\mathbb{F}^4$

  suppose $a_1, a_2, a_3, a_4 \in \mathbb{F}$ satisfy

$$(0,0,0,0) = a_1(1,0,0,0) + a_2(0,1,0,0) + a_3(0,0,1,0)$$

Then we have

$$(a_1,0,0,0) + (0,a_2,0,0) + (0,0,a_3,0) = (0,0,0,0)$$

$$(a_1, a_2, a_3, 0) = (0,0,0,0)$$

Equate the coordinates

$$a_1 = 0, \quad a_2 = 0, \quad a_3 = 0 \quad ✓$$

2.20 Example LINEARLY DEPENDENT LISTS

- $(2,3,1), (1,-1,2), (7,3,8)$ is linearly dependent in $\mathbb{F}^3$

Suppose $a_1, a_2, a_3 \in \mathbb{F}$ satisfy

$$a_1(2,3,1) + a_2(1,-1,2) + a_3(7,3,8) = (0,0,0)$$

$$(2a_1, 3a_1, a_1) + (a_2, -a_2, 2a_2) + (7a_3, 3a_3, 8a_3) = (0,0,0)$$

$$(2a_1 + a_2 + 7a_3,\ 3a_1 - a_2 + 3a_3,\ a_1 + 2a_2 + 8a_3) = (0,0,0)$$

$$\boxed{\begin{aligned} 2a_1 + a_2 + 7a_3 &= 0 \\ 3a_1 - a_2 + 3a_3 &= 0 \\ a_1 + 2a_2 + 8a_3 &= 0 \end{aligned}} \xRightarrow{\text{system solve}} \boxed{\begin{aligned} a_1 &= 2 \\ a_2 &= 3 \\ a_3 &= -1 \end{aligned}}$$

So at least one scalar is non-zero a_1, a_2, a_3

So $(2,3,1), (1,-1,2), (7,3,8)$ is linearly dependent in $\mathbb{F}^3$

- Every list of vectors in V containing the zero vector, such as

$$v_1, v_2, 0, v_4, v_5, v_6$$

is linearly dependent

Suppose $a_1, a_2, a_3, a_4, a_5, a_6 \in \mathbb{F}$ satisfy

$$a_1 v_1 + a_2 v_2 + a_3(0) + a_4 v_4 + a_5 v_5 + a_6 v_6 = 0$$

Then set

$$a_1 = 0, \quad a_2 = 0, \quad a_3 = 1, \quad a_4 = 0, \quad a_5 = 0, \quad a_6 = 0$$

So the scalar are NOT ALL ZEROs