Backward direction: If $U \cap W = \{0\}$, then $U+W$ is a direct sum.

Suppose $U \cap W = \{0\}$. Let $u \in U$, $w \in W$ satisfy $0 = u + w$.

But $0 = u + w$ implies $u = -w \in W$.

So $u \in U$ and $u \in W$; namely, $u \in U \cap W$.

So $u \in \{0\}$, and so $u = 0$.

And $0 = u + w$ with $u = 0$ implies $w = 0$.

So the only way to write 0 as a sum $u+w$ is to take $u=0$, $w=0$

By 1.44, $u+w$ is a direct sum.

2.A Span and Linear Independence

2.3 Definition: A linear combination of a list $v_1, \dots, v_m \in V$ is a vector of the form $a_1 v_1 + \dots + a_m v_m$ for some $a_1, \dots, a_m \in \mathbb{F}$.

Ex: Let $V = \mathbb{F}^3$. Is $(17, -4, 2)$ a linear combination of the list $(2,1,-3), (1,-2,4)$?

If yes, then there exist $a_1, a_2 \in \mathbb{F}$ such that $(17,-4,2) = a_1(2,1,-3) + a_2(1,-2,4)$

That is, we have $(17, -4, 2) = (2a_1, a_1, -3a_1) + (a_2, -2a_2, 4a_2)$

$$= (2a_1 + a_2,\ a_1 - 2a_2,\ -3a_1 + 4a_2).$$

Soln: $\boxed{\begin{aligned} 17 &= 2a_1 + a_2 \\ -4 &= a_1 - 2a_2 \end{aligned}} \Longrightarrow \boxed{a_1 = 6,\ a_2 = 5}$

$\boxed{2 = -3a_1 + 4a_2}$ $2 = -3(6) + 4(5)$

5 $5 \neq -3(6) + 4(5)$

Therefore, $(17, -4, 2)$ is a linear combination of $(2,1,-3), (1,-2,4)$,

but $(17,-4,5)$ is NOT a linear combination of $(2,1,-3), (1,-2,4)$.

2.5 Definition: The <u>span</u> of $v_1, \ldots, v_n$ is the set of all linear combinations of $v_1, \ldots, v_m \in V$, and is denoted

$$\text{span}(v_1, \ldots, v_m) = \{a_1 v_1 + \cdots + a_m v_m : a_1, \ldots, a_m \in \mathbb{F}\}.$$

↳ linear combination of $v_1, \ldots, v_m$.

(The span of the empty list ( ) is defined to be $\{0\}$.)

2-6 Example In $V = \mathbb{F}^3$,

- $(17, -4, 2)$ is a linear combination of $(2, 1, -3)$, $(1, -2, 4)$. Therefore, $(17, -4, 2) \in \text{span}((2,1,-3), (1,-2,4))$
- $(17, -4, 5)$ is NOT a linear combination of $(2,1,-3)$, $(1,-2,4)$. Therefore, $(17, -4, 5) \notin \text{span}((2,1,-3), (1,-2,4))$

2-7 Span is the smallest containing subspace

The span of a list of vectors $v_1, \ldots, v_m \in V$ is the smallest subspace of V containing $v_1, \ldots, v_m$.

Proof: First, we will prove that $\text{span}(v_1, \ldots, v_m)$ is a subspace of V.

- Additive identity: $0 = 0v_1 + \cdots + 0v_m \in \text{span}(v_1, \ldots, v_m)$
- Closed under addition: Let $a_1 v_1 + \cdots + a_m v_m$, $c_1 v_1 + \cdots + c_m v_m \in \text{span}(v_1, \ldots, v_m)$ for some $a_1, \ldots, a_m, c_1, \ldots, c_m \in \mathbb{F}$. Then we have $(a_1 v_1 + \cdots + a_m v_m) + (c_1 v_1 + \cdots + c_m v_m)$ $= (a_1 + c_1) v_1 + \cdots + (a_m + c_m) v_m \in \text{span}(v_1, \ldots, v_m)$.
- Closed under scalar multiplication: Let $\lambda \in \mathbb{F}$ be arbitrary. Then $\lambda(a_1 v_1 + \cdots + a_m v_m) = (\lambda a_1) v_1 + \cdots + (\lambda a_m) v_m \in \text{span}(v_1, \ldots, v_m)$

Therefore, $\text{span}(v_1, \ldots, v_m)$ is a subspace of V.

2.18 Example LINEARLY INDEPENDENT LISTS

- A list v_1 of one vector $v_1 \in V$ is linearly independent if and only if $v_1 \neq 0$. Then if $a_1 \in \mathbb{F}$ satisfies $a_1 v_1 = 0$, then $a_1 = 0$.

- A list $(1,0,0,0), (0,1,0,0), (0,0,1,0)$ is linearly independent in $\mathbb{F}^4$.

Suppose $a_1, a_2, a_3, a_4 \in \mathbb{F}$ satisfy $a_1(1,0,0,0) + a_2(0,1,0,0) + a_3(0,0,1,0) = (0,0,0,0)$.

Then we have $(a_1,0,0,0) + (0,a_2,0,0) + (0,0,a_3,0) = (0,0,0,0)$.

$$(a_1, a_2, a_3, 0) = (0,0,0,0)$$

Equate the coordinates: $a_1 = 0$, $a_2 = 0$, $a_3 = 0$ ✓

2.20 Example LINEARLY DEPENDENT LISTS

- $(2,3,1), (1,-1,2), (7,3,8)$ is linearly dependent in $\mathbb{F}^3$.

Suppose $a_1, a_2, a_3 \in \mathbb{F}$ satisfy $a_1(2,3,1) + a_2(1,-1,2) + a_3(7,3,8) = (0,0,0)$

$$(2a_1, 3a_1, a_1) + (a_2, -a_2, 2a_2) + (7a_3, 3a_3, 8a_3) = (0,0,0).$$

$$(2a_1 + a_2 + 7a_3,\ 3a_1 - a_2 + 3a_3,\ a_1 + 2a_2 + 8a_3) = (0,0,0).$$

$$\boxed{\begin{aligned} 2a_1 + a_2 + 7a_3 &= 0 \\ 3a_1 - a_2 + 3a_3 &= 0 \\ a_1 + 2a_2 + 3a_3 &= 0 \end{aligned}} \xrightarrow{\text{system-solve}} \boxed{\begin{aligned} a_1 &= 2 \\ a_2 &= 3 \\ a_3 &= -1 \end{aligned}}$$

So at least one scalar is non-zero in $a_1 a_2, a_3$.

So $(2,3,1), (1,-1,2), (7,3,8)$ is linearly dependent in $\mathbb{F}^3$.

- Every list of vectors in V containing the zero vector, such as $v_1, v_2, 0, v_4, v_5, v_6$, is linearly dependent.

Suppose $a_1, a_2, a_3, a_4, a_5, a_6 \in \mathbb{F}$ satisfy $a_1 v_1 + a_2 v_2 + a_3(0) + a_4 v_4 + a_5 v_5 + a_6 v_6 = 0$

Then set $a_1 = 0$, $a_2 = 0$, $a_3 = 1$, $a_4 = 0$, $a_5 = 0$, $a_6 = 0$.

So the scalars are NOT ALL ZERO.

Now, we will prove that span $(v_1, \dots, v_m)$ is the SMALLEST subspace of V.

First, notice that each v_j (for all $j = 1, \dots, m$) can be written as a linear combination of $v_1, \dots, v_m$. $v_j = 0v_1 + \dots + 0v_{j-1} + 1v_j + 0v_{j+1} + \dots + 0v_m \in \text{span}(v_1, \dots, v_m)$.

In other words, span $(v_1, \dots, v_m)$ contains each v_j, or equivalently span $(v_1, \dots, v_m)$ contains $v_1, \dots, v_m$.

Also, because every subspace of V is closed under scalar multiplication and addition, every subspace containing v_j contains all linear combination of $v_1, \dots, v_m$. In other words, every subspace contains span $(v_1, \dots, v_m)$.

This makes span $(v_1, \dots, v_m)$ the SMALLEST subspace of V.

2.8 Definition

If we have span $(v_1, \dots, v_m) = V$, then we say $v_1, \dots, v_m$ spans V.

2.9 Example The list $(1,0,0), (0,1,0), (0,0,1)$ spans $\mathbb{F}^3$.

Proof: Let $(x_1, x_2, x_3) \in \mathbb{F}^3$ be arbitrary.

Then we can write
$$(x_1, x_2, x_3) = (x_1, 0, 0) + (0, x_2, 0) + (0, 0, x_3)$$
$$= x_1(1,0,0) + x_2(0,1,0) + x_3(0,0,1)$$

So we have $(x_1, x_2, x_3) \in \text{span}((1,0,0), (0,1,0), (0,0,1))$.

In other words, $(1,0,0), (0,1,0), (0,0,1)$ spans $\mathbb{F}^3$.

2.11 Definition

- A function $p : \mathbb{F} \to \mathbb{F}$ is a polynomial if there exist $a_0, \ldots, a_m \in \mathbb{F}$ such that $p(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \cdots + a_m z^m$. for all $z \in \mathbb{F}$.
- The set of all polynomials with coefficients in $\mathbb{F}$ is called $P(\mathbb{F})$.

2.12

A polynomial $p \in P(\mathbb{F})$ is said to have degree m if there exists $a_0, a_1, \ldots, a_m \in \mathbb{F}$ with $a_m \neq 0$ such that $p(z) = a_0 + a_1 z + \cdots + a_m z^{(m)}$ for all $z \in \mathbb{F}$.

2.10 Definition

A vector space V is called finite-dimensional if there exists a list $v_1, \ldots, v_m \in V$ that spans V, that is $\text{span}(v_1, \ldots, v_m) = V$.

2.15 Definition

A vector space V is called infinite-dimensional if it is NOT finite dimensional.

Linear Independence

2.17 Definition

- A list $v_1, \ldots, v_m \in V$ is called linearly independent if the only choice of $a_1, \ldots, a_m \in \mathbb{F}$ that satisfies $a_1 v_1 + \cdots + a_m v_m = 0$ is $a_1 = 0, \ldots, a_m = 0$.

2.19 Definition

- A list $v_1, \ldots, v_m \in V$ is called linearly dependent if there exist $a_1, \ldots, a_m \in \mathbb{F}$, not all zero (in other words, some nonzero) such that $a_1 v_1 + \cdots + a_m v_m = 0$.

In other words, $v_1, \ldots, v_m \in V$ is NOT linearly independent.

2.A

Let's Revisit

2-7 Span is the smallest containing subspace

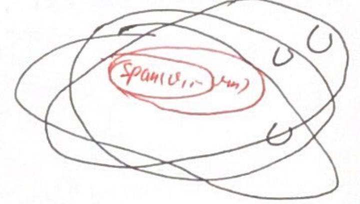
Every subspace of V containing $v_1, \ldots, v_m$ (each v_j, $j = 1, \ldots, m$) contains $\text{span}(v_1, \ldots, v_m)$

In other words: If a subspace contains $v_1, \ldots, v_m$, then it contains $\text{span}(v_1, \ldots, v_m)$.

Proof of that statement:

Let U be a subspace of V that contains $v_1, \ldots, v_m$. Since subspaces are closed under addition and scalar multiplication, $v_1, \ldots, v_m \in U$ implies $a_1 v_1 + \cdots + a_m v_m \in U$ for all $a_1, \ldots, a_m \in \mathbb{F}$. By definition, $\text{span}(v_1, \ldots, v_m) = \{a_1 v_1 + \cdots + a_m v_m : a_1, \ldots, a_m \in \mathbb{F}\}$.
Therefore, $\text{span}(v_1, \ldots, v_m) \subset U$.

In other words, every subspace of V that contains $v_1, \ldots, v_m \in V$ must also contain $\text{span}(v_1, \ldots, v_m)$. This makes $\text{span}(v_1, \ldots, v_m)$ the SMALLEST subspace of V that contains $v_1, \ldots, v_m$.



2-21 Linear Dependence Lemma

Suppose $v_1, \ldots, v_m$ is a linearly dependent list in V. Then there exists $j \in [1, 2, \ldots, m]$ such that:

(a) $v_j \in \text{span}(v_1, \ldots, v_{j-1})$
($v_j = a_1 v_1 + \cdots + a_{j-1} v_{j-1}$ for some $a_1, \ldots, a_{j-1} \in \mathbb{F}$)

(b) If the j^{th} term is removed from $v_1, \ldots, v_m$, (resulting in $v_1, \ldots, v_{j-1}, v_{j+1}, \ldots, v_m$)
then $\text{span}(v_1, \ldots, v_{j-1}, v_{j+1}, \ldots, v_m) = \text{span}(v_1, \ldots, v_m)$.

Proof of (a):

Since $v_1, \ldots, v_m$ is linearly dependent, there exist $a_1, \ldots, a_m \in \mathbb{F}$, not all 0, such that

$$a_1 v_1 + \cdots + a_m v_m = 0.$$

In particular, there exists $j \in \{1, \ldots, m\}$ such that $a_j \neq 0$.

So we have, $a_1 v_1 + \cdots + a_{j-1} v_{j-1} + a_j v_j + \cdots + a_m v_m = 0$.

Solve for v_j: $v_j = -\frac{a_1}{a_j} v_1 - \cdots - \frac{a_{j-1}}{a_j} v_{j-1}$.

In other words, v_j is a linear combination of $v_1, \ldots, v_{j-1}$.

Therefore, $v_j \in \operatorname{span}(v_1, \ldots, v_{j-1})$, proving (a).

Proof of (b):

Suppose $u \in \operatorname{span}(v_1, \ldots, v_m)$. Then there exist $c_1, \ldots, c_m \in \mathbb{F}$ such that $u = c_1 v_1 + \cdots + c_m v_m$.

Recall from the proof of part (a): $v_j = -\frac{a_1}{a_j} v_1 - \cdots - \frac{a_{j-1}}{a_j} v_{j-1}$.

We have

$$\begin{aligned} u &= c_1 v_1 + \cdots + c_m v_m \\ &= \boxed{c_1 v_1} + \cdots + \boxed{c_{j-1} v_{j-1}} + c_j v_j + c_{j+1} v_{j+1} + \cdots + c_m v_m \\ &= \boxed{c_1 v_1} + \cdots + \boxed{c_{j-1} v_{j-1}} + c_j \Big(\boxed{-\frac{a_1}{a_j} v_1} - \cdots \boxed{- \frac{a_{j-1}}{a_j} v_{j-1}} \Big) + c_{j+1} v_{j+1} + \cdots + c_m v_m \\ &= \Big(c_1 - c_j \frac{a_1}{a_j}\Big) v_1 + \cdots + \Big(c_{j-1} - c_j \frac{a_{j-1}}{a_j}\Big) v_{j-1} + c_{j+1} v_{j+1} + \cdots + c_m v_m \\ &\in \operatorname{span}(v_1, \ldots, v_{j-1}, v_{j+1}, \ldots, v_m) \end{aligned}$$

Therefore, $\operatorname{span}(v_1, \ldots, v_m) \subset \operatorname{span}(v_1, \ldots, v_{j-1}, v_{j+1}, \ldots, v_m)$.

On the other hand, Let $u \in \operatorname{span}(v_1, \ldots, v_{j-1}, v_{j+1}, \ldots, v_m)$.

Then there exist $a_1, \ldots, a_m \in \mathbb{F}$ such that $u = a_1 v_1 + \cdots + a_{j-1} v_{j-1} + a_{j+1} v_{j+1} + \cdots + a_m v_m$.

But also, $u = a_1 v_1 + \cdots + a_{j-1} v_{j-1} + \boxed{0 v_j} + a_{j+1} v_{j+1} + \cdots + a_m v_m \in \operatorname{span}(v_1, \ldots, v_m)$

So $\operatorname{span}(v_1, \ldots, v_{j-1}, v_{j+1}, \ldots, v_m) \subset \operatorname{span}(v_1, \ldots, v_m)$.

Therefore, we have the set equality $\operatorname{span}(v_1, \ldots, v_{j-1}, v_{j+1}, \ldots, v_m) = \operatorname{span}(v_1, \ldots, v_m)$, proving part (b).

2.24 Example Is the list $(1,2,3),(4,5,8),(9,6,7),(-3,2,8)$ linearly independent in $\mathbb{R}^3$?. No.

The list $(1,0,0),(0,1,0),(0,0,1)$ spans $\mathbb{R}^3$. This list has length 3.

- No list of length greater than 3 ~~spans $\mathbb{R}^3$~~ is linearly independent in $\mathbb{R}^3$, because any vector in the list $(1,2,3),(4,5,8),(9,6,7),(-3,2,8)$ is a linear combination of the other three.

2.25 Example Does the list $(1,2,3,-5),(4,5,8,3),(9,6,7,-1)$ span $\mathbb{R}^4$?

No. The list $(1,0,0,0),(0,1,0,0),(0,1,0,0),(0,0,1,0),(0,0,0,1)$ spans $\mathbb{R}^4$.

No list of length less than 4 spans $\mathbb{R}^4$.

2.26 Finite dimensional subspaces

Every subspace of a finite-dimensional vector space is finite dimensional.

We're skipping the proof for our lecture.

2.B Bases

2.27 Definition

A basis of V is a list of vectors $v_1, \ldots, v_m$ of V that is linearly independent and spans V.

2.28 Example

- The list $(1,0,\ldots,0),(0,1,0\ldots,0),\ldots,(0,\ldots,0,1)$ is a basis of $\mathbb{F}^n$.
  It is called the standard basis of $\mathbb{F}^n$.
- $(1,0),(0,1)$ is a basis of $\mathbb{F}^2$.
  (standard basis)
- $(1,2),(3,5)$ is a basis of $\mathbb{F}^2$.
- $(2,3),(4,6)$ is NOT a basis of $\mathbb{F}^2$.
  (It is linearly dependent)