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2.7 $\boxed{\text{Edit}}$ Span is the smallest containing subspace

- Every subspace of V containing $v_1, \dots v_m$ contains span $(v_1, \dots v_m)$.

In other words:

If a subspace contains $v_1, \dots v_m$, then it contains span $(v_1, \dots v_m)$

Proof:

Let U be a subspace of V that contains $v_1, \dots v_m$. Since subspaces are closed under under addition and scalar multiplication, $v_1, \dots v_m \in U$ implies

$$a_1 v_1 + \dots + a_m v_m \in U$$

$\forall a_1, \dots, a_m \in \mathbb{F}$ By Definition,

$$\text{span}(v_1, \dots v_m) = \{a_1 v_1 + \dots + a_m : a_1, \dots a_m \in \mathbb{F}\}.$$

Therefore, $\text{span}(v_1, \dots, v_m) \subset U$.

In other words, every subspace of V that contains $v_1, \dots, v_m \in V$ must also contain span$(v_1, \dots, v_m)$. This makes span$(v_1, \dots, v_m)$ the smallest subspace of V that contains $v_1, \dots, v_m$.

2.21 Linear Dependence Lemma

Suppose $v_1, \ldots, v_m$ is a linearly dependent list in V. Then there exists $j \in \{1, 2, \ldots, m\}$ such that:

a) $v_j \in \operatorname{span}(v_1, \ldots v_{j-1})$

$v_j = a_1 v_1 + \ldots + a_{j-1} v_{j-1}$ for some $a_1 \ldots a_{j-1} \in \mathbb{F}$

b) if the j-th term's removed from $v_1, \ldots v_m$, (resulting in $v_1, \ldots, v_{j-1}, v_{j+1}, \ldots, v_m$) then $\operatorname{span}(v_1, \ldots v_{j-1}, v_{j+1}, \ldots v_m) = \operatorname{span}(v_1, \ldots v_m)$

Proof a):

Since $v_1, \ldots v_m$ is linearly dependent, there exists $a_1, \ldots a_m \in \mathbb{F}$, not all 0, such that $a_1 v_1 + \ldots + a_m v_m = 0$

In particular, there exists $j \in \{1, \ldots, m\}$ such that $a_j \neq 0$.

So we have:

$$a_1 v_1 + \ldots + a_{j-1} v_{j-1} + a_j v_j + \ldots + a_m v_m = 0$$

Solve for v_j:

$$v_j = -\frac{a_1}{a_j} v_1 - \ldots - \frac{a_{j-1}}{a_j} v_{j-1}$$

In other words, v_j is a linear combination of $v_1, \ldots, v_{j-1}$.

Therefore, $v_j \in \operatorname{Span}(v_1, \ldots v_{j-1})$, proving a).

Proof b):

Suppose $u \in \operatorname{span}(v_1, \ldots v_m)$. Then there exist $c_1, \ldots c_m \in \mathbb{F}$ such that $u = c_1 v_1 + \ldots + c_m v_m$.

Recall from the proof of part a):

$$v_j = -\frac{a_1}{a_j} v_1 - \ldots - \frac{a_{j-1}}{a_j} v_{j-1}$$

We have:

$$u = c_1 v_1 + \ldots + c_m v_m$$
$$= c_1 v_1 + \ldots + c_{j-1} v_{j-1} + c_j v_j + c_{j+1} v_{j+1} + \ldots + c_m v_m.$$

$= c_1 v_1 + \dots + c_{i-1} v_{i-1} + c_i \left(-\frac{a_1}{a_i} v_1 - \dots - \frac{a_{j-1}}{a_i} v_{j-1} \right) + c_{i+1} v_{i+1} + \dots + c_m v_m$

$= \left(c_1 - c_i \frac{a_1}{a_i}\right) v_1 + \dots + \left(c_{i-1} - c_i \frac{a_{i-1}}{a_i}\right) v_{j-1} + c_{i+1} v_{j+1} + \dots + c_m v_m \in \text{span}(v_1, \dots, v_{j-1}, v_{j+1}, \dots v_m)$

Therefore, $\text{span}(v_1, \dots, v_m) \subset \text{span}(v_1, \dots, v_{j-1}, v_{j+1}, \dots v_m)$.

On the other hand, let $u \in \text{span}(v_1, \dots, v_{j-1}, v_{j+1}, \dots v_m)$

Then there exist $a_1, \dots a_m \in \mathbb{F}$ such that

$$u = a_1 v_1 + \dots + a_{j-1} v_{j-1} + a_{j+1} v_{j+1} + \dots + a_m v_m.$$

But also,

$$u = a_1 v_1 + \dots + a_{j-1} v_{j-1} + 0 v_j + a_{j+1} v_{j+1} + \dots + a_m v_m$$

$\in \text{span}(v_1, \dots v_m)$.

So $\text{span}(v_1, \dots, v_{j-1}, v_{j+1}, \dots v_m) \subset \text{span}(v_1, \dots v_m)$.

Therefore, we have the set equality $\text{span}(v_1, \dots v_{j-1}, v_{j+1}, \dots v_m) = \text{span}(v_1, \dots v_m)$ proving part b).

2.24 Example Is the list $(1,2,3), (4,5,8), (9,6,7), (-3,2,8)$ linearly dependent in $\mathbb{R}^3$? NO

The list $(1,0,0), (0,1,0), (0,0,1)$ spans $\mathbb{R}^3$.

The list has length 3.

No list of length greater than 3 is linearly independent in $\mathbb{R}^3$ because any vector in the list $(1,2,3), (4,5,8), (9,6,7), (-3,2,8)$ is linear combination of the other three.

2.25 Example Does the list $(1,2,3,-5), (4,5,8,3), (9,6,7,-1)$ span $\mathbb{R}^4$? NO

The list $(1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1)$ spans $\mathbb{R}^4$

No list of length less than 4 spans $\mathbb{R}^4$.

2.26 Finite dimensional subspaces

Every subspace of a finite-dimensional vector space is finite dimensional. *Proof is in textbook.

2.B Bases

2.27 [Def]

A bases of V is a list of vectors $v_1, \dots, v_m$ of V that is linearly dependent and spans V.

2.28 Example

The list $(1, 0, \dots, 0), (0, 1, 0, \dots, 0), \dots, (0, \dots, 0, 1)$ is a bases of $\mathbb{F}^n$.

It is called the standard basis of $\mathbb{F}^n$.

- $(1, 0), (0, 1)$ is a basis of $\mathbb{F}^2$.
- $(1, 2), (3, 5)$ is a basis of $\mathbb{F}^2$.
- $(2, 3), (4, 6)$ is NOT a basis of $\mathbb{F}^2$. ← linearly dependent
- $(1, 2, 3), (2, 1, 3)$ is Not a basis of $\mathbb{F}^3$
- $(1, 2), (3, 5), (4, 13)$ is NOT a basis of $\mathbb{F}^2$ ← l.d.
- $(1, 1, 0), (0, 0, 1)$ is a basis of $\{(x, x, y) \in \mathbb{F}^3 : x, y \in \mathbb{F}\}$

because $(x, x, y) = (x, x, 0) + (0, 0, y) = x(1, 1, 0) + y(0, 0, 1)$

- $(1, -1, 0), (1, 0, -1)$ is a basis of $\{(x, y, z) \in \mathbb{F}^3 : x + y + z = 0\}$

because $x + y + z = 0$ implies $z = -x - y$

2.29 Criterion for Bases

A list $v_1, \dots, v_n \in V$ is a basis of V IFF every $v \in V$ can be written uniquely (in only one way) in the form.

$$v = a_1 v_1 + \dots + a_n v_n \qquad \text{for some } a_1, \dots, a_n \in \mathbb{F}$$

Proof: Forward Direction

Suppose $v_1, \dots, v_n$ is a basis of V. $\forall v \in V$, the exists $a_1, \dots, a_n$ such that,

① $v = a_1 v_1 + \dots + a_n v_n$.

We need to show $c_1, \dots, c_n \in \mathbb{F}$ also satisfy

② $v = c_1 v_1 + \dots + c_n v_n$

① − ② $= 0 = (a_1 - c_1) v_1 + \dots + (a_n - c_n) v_n$

So $a_1 - c_1 = 0, \dots, a_n - c_n = 0$
Therefore, $a_1 = c_1, \dots, a_n = c_n$
and so the representation is unique.

Backward direction

Suppose we can write every $v \in V$ uniquely in the form: $v = a_1 v_1 + \dots + a_n v_n$.

Then V is a linear combination of $v_1, \dots, v_n$, which means $v_1, \dots, v_n$ spans V.

Now we must prove that $v_1, \dots, v_n$ is linearly independent. Suppose $a_1, \dots, a_n \in \mathbb{F}$ satisfy

$$a_1 v_1 + \dots + a_n v_n = 0$$

Since the representation of every $v \in V$ ($v = 0$ in particular) is unique, we must have $a_1 = 0, \dots, a_n = 0$.

Therefore, $v_1, \dots, v_n$ is linearly independent.

Since we proved that $v_1, \dots, v_n$ is a linearly independent set that spans V, we can conclude that $v_1, \dots, v_n$ is a basis of V.

2.31 Spanning list contains a basis

Every spanning list in a vector space can be reduced to a basis of the vector space.

For example,

$$(1,0), (0,1), (2,3)$$

is a spanning list of $\mathbb{F}^2$. But we can reduce this list to $(1,0), (0,1)$.

2.32 Basis of finite-dimensional vector space

Every finite-dimensional vector space has a basis.

Proof: According to 2.10, there exists a spanning list of every finite-dimensional vector space. By 2.31, the spanning list can be reduced to a basis.

2.33 linearly independent list extends to basis
Let V be a finite-dimensional vector space, and let $u_1, \dots, u_m \in V$ be a linearly independent list. Then this list can be extended to a basis $w_1, \dots, w_n$ of V.

Proof: Suppose $u_1, \dots, u_m \in V$ is linearly independent and $w_1, \dots, w_n \in V$ is a basis.
Then the list,

$$u_1, \dots, u_m, w_1, \dots, w_n$$

spans V, Apply the procedure of 2.31 of Axler to remove some vectors from $u_1, \dots, u_m, w_1, \dots, w_n$ to reduce to this list a basis of V.

For example, the list $(2,3,4), (9,6,8)$ is linearly independent in $\mathbb{F}^3$.
Then follow proof 2.33,

$$(2,3,4), (9,6,8), (0,1,0),$$

which is a basis of $\mathbb{F}^3$.

2.34 Every subspace of V is a part of a direct sum $= V$

Suppose V is finite-dimensional vector space and U is a subspace of V. Then there exists a subspace W of V such that $V = U \oplus W$
Proof: since V is finite-dimensional and U is a subspace of V, by 2.26, U is also finite dimensional.
By 2.32, there exists a basis $u_1, \dots, u_m$ of U.
This means $u_1, \dots, u_m$ is linearly independent in V.
By 2.33, we can extend $u_1, \dots, u_m$ to a basis $u_1, \dots, u_m, w_1, \dots, w_m$ of V. Now let $W = \text{span}(w_1, \dots, w_m)$
To prove $V = U \oplus W$; By 1.45, we need to prove

$V = U + W$ & $U \cap W = \{0\}$

First, we prove $V = U + W$

Suppose $v \in V$. Since $u_1, \dots, u_m, w_1, \dots, w_n$ spans V. there exist $a_1, \dots, a_m, b_1, \dots, b_n \in \mathbb{F}$ such that

$v = a_1 u_1 + \dots + a_m u_m + b_1 w_1 + \dots + b_n w_n$

So $v = u + w$, where $u \in U$ & $w \in W$. So $V = U + W$

Proof: $U \cap W = \{0\}$

Suppose $v \in U \cap W$ $\exists$ $a_1, \dots, a_m, b_1, \dots, b_n \in \mathbb{F}$ such that $v = a_1 u_1 + \dots + a_m u_m$

$= b_1 w_1 + \dots + b_n w_n$

$a_1 u_1 + \dots + a_m u_m - b_1 w_1 - \dots - b_n w_n = 0$

Since $u_1, \dots, u_m, \dots, w_n$ are linearly independent, all the scalars are zero,

$a_1 = 0, \dots, a_m = 0, b_1 = 0, \dots, b_n = 0$.

Therefore, $v = a_1 u_1 + \dots + a_m u_m = 0u_1 + \dots + 0u_m = 0 \in \{0\}$

& $v = b_1 w_1 + \dots + b_n w_n = 0w_1 + \dots + 0w_n = 0 \in \{0\}$

Therefore, $U \cap W \subset \{0\}$

So $U \cap W = \{0\}$, as desired. ☒