2.7 Span is the smallest containing subspace.

Every subspace of V containing $v_1, \ldots, v_m$ contains $\text{span}(v_1, \ldots, v_m)$

In other words,

If a subspace contains $v_1, \ldots, v_m$, then it contains $\text{span}(v_1, \ldots, v_m)$.

In other words, every subspace of V that contains $v_1, \ldots, v_m \in V$ must also contain $\text{span}(v_1, \ldots, v_m)$. This makes $\text{span}(v_1, \ldots, v_m)$ the smallest subspace of V that contains $v_1, \ldots, v_m$.

Proof:

Let U be a subspace of V that contains $v_1, \ldots, v_m$.

Since subspaces are closed under addition and scalar multiplication, $v_1, \ldots, v_m \in U$ implies $a_1 v_1 + \cdots + a_m v_m \in U$ for all $a_1, \ldots, a_m \in \mathbb{F}$. By definition,

$$\text{span}(v_1, \ldots, v_m) = \{a_1 v_1 + \cdots + a_m v_m : a_1, \ldots, a_m \in \mathbb{F}\}$$

$\therefore \text{span}(v_1, \ldots, v_m) \subset U$.

2.21 Linear Dependence Lemma.

Suppose $v_1, \dots, v_m$ is a linearly dependent list in V. Then there exists $j \in \{1, \dots, m\}$ such that:

a). $v_j \in \operatorname{span}(v_1, \dots, v_{j-1})$

($v_j = a_1 v_1 + \cdots + a_{j-1} v_{j-1}$ for some $a_1, \dots, a_{j-1} \in \mathbb{F}$)

b). If the j^{th} term is removed from $v_1, \dots, v_m$,

(resulting in $v_1, \dots, v_{j-1}, v_{j+1}, \dots, v_m$)

then $\operatorname{span}(v_1, \dots, v_{j-1}, v_{j+1}, \dots, v_m) = \operatorname{span}(v_1, \dots, v_m)$

Proof of a):

since $v_1, \dots, v_m$ is linearly dependent, there exist $a_1, \dots, a_m \in \mathbb{F}$, not all 0, such that

$$a_1 v_1 + \dots + a_m v_m = 0$$

In particular, there exists $j \in \{1, \dots, m\}$ such that $a_j \neq 0$.

So we have,

$a_1 v_1 + \dots + a_{j-1} v_{j-1} + a_j v_j + \dots + a_m v_m = 0.$

Solve for v_j:

$$v_j = -\frac{a_1}{a_j} v_1 - \dots - \frac{a_{j-1}}{a_j} v_{j-1}.$$

In other words, v_j is a linear combination of $v_1, \dots, v_{j-1}$.

Therefore, $v_j \in \operatorname{span}(v_1, \dots, v_{j-1})$, proving (a).

Proof of b):

Suppose $u \in \text{span}(v_1, \ldots, v_m)$. Then there exist $c_1, \ldots, c_m \in \mathbb{F}$ such that $u = c_1v_1 + \ldots + c_mv_m$.

Recall from the proof of part (a):

$$v_j = -\frac{a_1}{a_j}v_1 - \ldots - \frac{a_{j-1}}{a_j}v_{j-1}$$

We have

$$\begin{aligned} u &= c_1v_1 + \ldots + c_mv_m \\ &= c_1v_1 + \ldots + c_{j-1}v_{j-1} + c_jv_j + \ldots + c_mv_m \\ &= c_1v_1 + \ldots + c_{j-1}v_{j-1} + c_j\left(-\frac{a_1}{a_j}v_1 - \cdots - \frac{a_{j-1}}{a_j}v_{j-1}\right) + c_{j+1}v_{j+1} + \cdots + c_mv_m \\ &= \left(c_1 - c_j\frac{a_1}{a_j}\right)v_1 + \cdots + \left(c_{j-1} - c_j\frac{a_{j-1}}{a_j}\right)v_{j-1} + c_{j+1}v_{j+1} + \ldots + c_mv_m \\ &\in \text{span}(v_1, \ldots, v_{j-1}, v_{j+1}, \ldots, v_m) \end{aligned}$$

Therefore, $\text{span}(v_1, \ldots, v_m) \subset \text{span}(v_1, \ldots, v_{j-1}, v_{j+1}, \ldots, v_m)$

In other words, let $u \in \text{span}(v_1, \ldots, v_{j-1}, v_{j+1}, \ldots, v_m)$

Then there exist $a_1, \ldots, a_m \in \mathbb{F}$ such that

$$u = a_1v_1 + \cdots + a_{j-1}v_{j-1} + a_{j+1}v_{j+1} + \ldots + a_mv_m$$

But also,

$$\begin{aligned} u &= a_1v_1 + \ldots + a_{j-1}v_{j-1} + 0v_j + a_{j+1}v_{j+1} + \ldots + a_mv_m \\ &\in \text{span}(v_1, \ldots, v_m) \end{aligned}$$

So $\text{span}(v_1, \ldots, v_{j-1}, v_{j+1}, \ldots v_m) \subset \text{span}(v_1, \ldots, v_m)$

Therefore, we have the set equality

$$\text{span}(v_1, \ldots, v_{j-1}, v_{j+1}, \ldots, v_m) = \text{span}(v_1, \ldots, v_m)$$

proving part (b).

2.24 Example

Is the list $(1,2,3)$, $(4,5,8)$, $(2,6,7)$, $(-3,2,8)$ linearly independent in $\mathbb{R}^3$?

The list $(1,0,0)$, $(0,1,0)$, $(0,0,1)$ spans $\mathbb{R}^3$.

This list has length 3. No list of length greater than 3 span $\mathbb{R}^3$.

Because the fourth vector in the list

$(1,2,3)$, $(4,5,8)$, $(2,6,7)$, $(-3,2,8)$

is a linear combination of the other three.

2.25 Example

Does the list $(1,2,3,-5)$, $(4,5,8,3)$, $(2,6,7,-1)$ span $\mathbb{R}^4$?

No, the list $(1,0,0,0)$, $(0,1,0,0)$, $(0,0,1,0)$, $(0,0,0,1)$ spans $\mathbb{R}^4$.

No list of length less than 4 spans $\mathbb{R}^4$.

2.26 Finite dimensional subspaces.

Every subspace of a finite-dimensional vector space is finite dimensional

(Proof in textbook).

2.27 Definition

A basis of V is a list of vectors $v_1, \dots, v_m$ of V that is linearly independent and spans V.

2.28 Example

- The list $(1,0,\dots,0)$, $(0,1,0,\dots,0)$, $\dots$, $(0,\dots,0,1)$ is a basis of $\mathbb{F}^n$.

  It is called the standard basis of $\mathbb{F}^n$.

- $(1,0),(0,1)$ is a basis of $\mathbb{F}^2$. (standard basis)
- $(1,2),(3,5)$ is a basis of $\mathbb{F}^2$.
- $(2,3),(4,6)$ is not a basis of $\mathbb{F}^2$. (It's linearly independent).
- $(1,2,3),(2,1,3)$ is not a basis of $\mathbb{F}^3$. (Only two vectors in $\mathbb{F}^3$, therefore does not span $\mathbb{F}^3$)
- $(1,2),(3,5),(4,13)$ is not a basis of $\mathbb{F}^2$ (linearly dependent)
- $(1,1,0),(0,0,1)$ is a basis of $\{(x,x,y)\in\mathbb{F}^3 : x,y\in\mathbb{F}\}$ because $(x,x,y) = x(1,1,0)+y(0,0,1)$
- $(1,-1,0),(1,0,-1)$ is a basis of $\{(x,y,z)\in\mathbb{F}^3 : x+y+z=0\}$ because $x+y+z=0$ implies $z=-x-y$.

2.29 Criterion for basis.

A list $v_1,\dots,v_n\in V$ is a basis of V iff every $v\in V$ can be written uniquely (in only one way) in the form

$$v = a_1v_1+\dots+a_nv_n$$

for some $a_1,\dots,a_n\in\mathbb{F}$.

Proof (forward direction)

Suppose $v_1,\dots,v_n$ is a basis of V. For all $v\in V$, there exist $a_1,\dots,a_n$ such that

$$v = a_1v_1+\dots+a_nv_n$$

We need to show that this representation is unique.

Suppose $c_1,\dots,c_n\in\mathbb{F}$ also satisfy

$$v = c_1v_1+\dots+c_nv_n$$

Substracting $\star - \star$, We get

$$0 = (a_1 - c_1)v_1 + \dots + (a_n - c_n)v_n$$

Since 0 can be written in only one way, we must have

$$a_1 - c_1 = 0, \dots, a_n - c_n = 0$$

Therefore, $a_1 = c_1, \dots, a_n = c_n$

and so the representation is unique.

(Backward direction)

Suppose we can write every $v \in V$ uniquely in the form

$$v = a_1 v_1 + \cdots + a_n v_n$$

Then v is a linear combination of $v_1, \dots, v_n$, which means $v_1, \dots, v_n$ spans V.

Now we must prove that $v_1, \dots, v_n$ is linearly independent

Suppose $a_1, \dots, a_n \in \mathbb{F}$ satisfy $a_1 v_1 + \dots + a_n v_n = 0 \Rightarrow v = 0$

Since the representation of every $v \in V$ ($v = 0$, in particular) is unique, we must have $a_1 = 0, \dots, a_n = 0$.

Therefore, $v_1, \dots, v_n$ is linearly independent.

Since we proved that $v_1, \dots, v_n$ is a linearly independent set that spans V, we conclude that $v_1, \dots, v_n$ is a basis of V.

2.31 Spanning list contains a basis.

Every spaning list in a vector space can be reduced to a basis of the vector space.

For example,

$(1, 0)$ $(0, 1)$ $(2, 3)$

is a spanning list of $\mathbb{F}^2$, But we can reduce this list to $(1,0), (0,1)$.

2.32 Basis of a finite-dimensional vector space.

Proof: According to 2.10 of Axler, there exists a spanning list of every finite-dimensional vector space. By 2.31, the spanning list can be reduced to a basis.

2.33 Linearly independent list extends to a basis.

Let V be a finite-dimensional vector space, and let $u_1, \dots, u_m \in V$ be a linearly independent list. Then this list can be extended to a basis $w_1, \dots, w_n$ of V.

Proof: Suppose $u_1, \dots, u_m \in V$ is linearly independent and $w_1, \dots, w_n \in V$.

Then the list $u_1, \dots, u_m, w_1, \dots, w_n$ spans V, Apply the procedure of 2.31 of Axler to remove some vectors from $u_1, \dots, u_m, w_1, \dots, w_m$ (if necessary) to reduce this spanning list to a basis of V.

For example, the list $(2,3,4), (9,6,8)$ is linearly independent in $\mathbb{F}^3$, Then following the procedure of the proof of 2.33 Axler, we can obtain a list.

$$(2,3,4), (9,6,8), (0,1,0),$$

which is a basis of $\mathbb{F}^3$.

2.34 Every subspace of V is part of a direct sum equal to V.

Suppose V is a finite-dimensional vector space, and V is a subspace of V. Then there exists a subspace W of V such that $V = U \oplus W$. (can write $v = u + w$ only one way $v \in V$, $v \in U$, $w \in W$)

Proof: Since V is finite-dimensional and V is a subspace of V, by 2.26 of Axler, U is also finite-dimensional. By 2.32 of Axler, there exists a basis $u_1, \dots, u_m$ of U. This means in particular that $u_1, \dots, u_m$ is linearly independent in V. By 2.33 Axler, we can extend $u_1, \dots, u_m$ to a basis $u_1, \dots, u_m, w_1, \dots, w_n$ of V.

Now, let $W = \operatorname{span}(w_1, \dots, w_n)$

To prove $V = U \oplus W$. By 1.45 of Axler, we just need to prove $V = U + W$ and $U \cap W = \{0\}$

First, we will prove $V = U + W$.

Suppose $v \in V$, Since $u_1, \dots, u_m, w_1, \dots, w_n$ spans V there exist $a_1, \dots, a_m, b_1, \dots, b_n \in \mathbb{F}$ such that

$$V = \underbrace{a_1 u_1 + \cdots + a_m v_m}_{U} + \underbrace{b_1 v_1 + \cdots + b_n v_n}_{W}$$

So $v = u + w$, where $u \in U$ and $w \in W$. So $V = U + W$.

Proof: Next we will show $U \cap W = \{0\}$

Suppose $v \in U \cap W$. Then $u \in V$ and $u \in W$. so there exist $a_1, \dots, a_m, b_1, \dots, b_n \in \mathbb{F}$ s.t.

$V = a_1 u_1 + \cdots + a_m u_m = b_1 w_1 + \cdots + b_n w_n$

Substracting, we get:

$a_1u_1 + \cdots + a_mu_m - b_1w_1 - \cdots - b_nw_n = 0$

Since $u_1, \ldots, u_m, w_1, \ldots, w_n$ are L.I., all the scalars are zero.

$a_1 = 0, \ldots, a_m = 0, -b_1 = 0, \ldots, -b_n = 0$

Therefore, $v = a_1u_1 + \cdots + a_mu_m$

$= 0u_1 + \cdots + 0u_m$

$= 0 \in \{0\}$

and $v = b_1w_1 + \cdots + b_nw_n$

$= 0w_1 + \cdots + 0w_n$

$= 0 \in \{0\}$

Therefore, $U \cap W \subseteq \{0\}$

So $U \cap W = \{0\}$, as desired.