2.B Bases

2.27 Definition

A <u>basis</u> of V is a list of vectors $v_1, \dots, v_m$ of V that is linearly independent and spans V.

2.28 Example

- The list $(1,0,\dots,0), (0,1,0,\dots,0), \dots, (0,\dots,0,1)$ is a basis of $\mathbb{F}^n$.
  It is called the <u>standard basis of $\mathbb{F}^n$</u>.

- ~~[illegible]~~ $(1,0), (0,1)$ is a standard basis of $\mathbb{F}^2$.
- $(1,2), (3,5)$ is a basis of $\mathbb{F}^2$.
- $(2,3), (4,6)$ is NOT a basis of $\mathbb{F}^2$.
  (It is linearly dependent)

- $(1,2,3), (2,1,3)$ is NOT a basis of $\mathbb{F}^3$.
  (only two vectors in $\mathbb{F}^3$, therefore does not span $\mathbb{F}^3$)
- $(1,2), (3,5), (4,13)$ is NOT a basis of $\mathbb{F}^2$.
  (three vectors in $\mathbb{F}^2$, therefore one vector is linear combination of the other two)
  (linearly dependent)

- $(1,1,0), (0,0,1)$ is a basis of $\{(x,x,y) \in \mathbb{F}^3 : x, y \in \mathbb{F}\}$
  because
$$(x,x,y) = (x,x,0) + (0,0,y)$$
$$= x(1,1,0) + y(0,0,1)$$

• $(1,-1,0), (1,0,-1)$ is a basis of $\{(x,y,z) \in \mathbb{F}^3 : x+y+z = 0\}$
because $x+y+z=0$ implies $z = -x-y$ so $\}U$

$(x,y,z) = (x,y,-x-y)$

$$\begin{aligned}(x,y,z) &= (x,y,-x-y)\\ &= (x,0,-x) + (0,y,-y)\\ &= x(1,0,-1) + y(0,1,-1)\end{aligned}$$

So $(1,0,-1), (0,1,-1)$ spans U.

Suppose $a,b \in \mathbb{F}$ satisfy

$$a(1,0,-1) + b(0,1,-1) = (0,0,0)$$
$$(a,0,-a) + (0,b,-b) = (0,0,0)$$
$$(a,b,-a-b) = (0,0,0)$$

$a=0$

$b=0$

$-a-b=0$

$0=0$ ✓

Therefore the list $(1,0,-1),(0,1,-1)$ is linearly independent.

So $(1,0,-1), (0,1,-1)$ is a basis of U.

• The list $1, z, \ldots, z^m$ is a basis of $\mathcal{P}_m(\mathbb{F})$.
Every polynomial $p \in \mathcal{P}_m(\mathbb{F})$ is written $p(z) = a_0 + a_1 z + \ldots + a_m z^m$ for all $z \in \mathbb{F}$ for all $a_0, a_1, a_2, \ldots, a_m \in \mathbb{F}$. Notice that $p(z)$ is a lin. combo of $1, z, z^2, \ldots, z^m$

Continue after 2.34

2.29 Criterion for basis

A list $v_1, .., v_n \in V$ is a basis of V if and only if every $v \in V$ can be written uniquely (in only one way) in the form

$$v = a_1 v_1 + .. + a_n v_n$$

for some $a_1, .., a_n \in \mathbb{F}$

Proof Forward direction

Suppose $v_1, .., v_n$ is a basis of V. For all $v \in V$, there exist $a_1, .., a_n$ such that

$$\star \; v = a_1 v_1 + .. + a_n v_n$$

We need to show that this representation is unique.

Suppose $c_1, .., c_n \in \mathbb{F}$ also satisfy

$$\star \; v = c_1 v_1 + .. + c_n v_n.$$

Subtracting $\star - \star$, we get

$$0 = (a_1 - c_1) v_1 + .. + (a_n - c_n) v_n.$$

Since 0 can be written in only one way, we must have

$$a_1 - c_1 = 0, .., a_n - c_n = 0.$$

Therefore,

$$a_1 = c_1, \dots, a_n = c_n$$

and so the representation is unique.

Backward direction

Suppose we can write every $v \in V$ uniquely into form
$$v = a_1 v_1 + \dots + a_n v_n.$$

Then v is a linear combination of $v_1, \dots, v_n$, which means $\boxed{v_1, \dots, v_n \text{ spans } V.}$

Now we must prove that $v_1, \dots, v_n$ is linearly independent.
Suppose $a_1, \dots, a_n \in \mathbb{F}$ satisfy
$$a_1 v_1 + \dots + a_n v_n = 0. \quad v=0$$
Since the representation of every $v \in V$ ($v=0$, in particular), is unique, we must have
$$a_1 = 0, \dots, a_n = 0.$$
Therefore, $v_1, \dots, v_n$ is $\boxed{\text{linearly independent}}$

Since, we proved that $v_1, \dots, v_n$ is a linearly independent set that spans V, we conclude that $v_1, \dots, v_n$ is a basis of V. ∎

2.31 Spanning list contains a basis

Every spanning list in a vector space can be reduced to a basis of the vector space.
For Ex, → basis of $\mathbb{F}^2$
$\boxed{(1,0), (0,1)}, (2,3)$ is a spanning list of $\mathbb{F}^2$. But we can reduce this list to
$$(1,0), (0,1)$$
(Proof skipped for 2.31)

2.32 Basis of finite-dimensional vector space

Every finite-dimensional vector space has a basis.

Proof: According to 2.10 of Axler (definition of finite-dimensional vector space), there exists a spanning list of every finite-dimensional vector space. By 2.31, the spanning list can be reduced to a basis. $\boxtimes$

2.33 Linearly independent list extends to a basis

Let V be a finite-dimensional vector space, and let $u_1, \dots, v_m \in V$ be a linearly independent list. Then this list can be extend to a basis $w_1, \dots, w_n$ of V.

Proof: Suppose ~~the~~ $u_1, \dots, u_m \in V$ is linearly independent and $w_1, \dots, w_n \in V$ is a basis.

Then the list

$$u_1, \dots, u_m, w_1, \dots, w_n$$ — extend to a spanning list of V

~~sp~~ spans V. Apply the procedure of 2.31 of Axler to remove some vectors from $u_1, \dots, u_m, w_1, \dots, w_n$ (if necessary) to reduce this spanning list to a basis of V. $\boxtimes$

For ex, the list $(2,3,4), (9,6,8)$ is linearly independent in $\mathbb{F}^3$. Then following the procedure of the proof of 2.33 of Axler, we can obtain a list

$$(2,3,4), (9,6,8), (0,1,0)$$

which is a basis of $\mathbb{F}^3$.

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2.34 Every subspace of V is part of a direct sum equal to V.

Suppose V is finite-dimensional vector space, and U is a subspace of V. Then there exists a subspace W of V such that $V = U \oplus W$. 1

can write $v = u + w$ in only one way
$v \in V$, $u \in U$, $w \in W$

Proof: Since V is finite dimensional and U is a subspace of V, by 2.26 of Axler, U is also finite-dimensional. By 2.32 of Axler, there exists a basis $u_1, \ldots, u_m$ of U. This means in particular that $u_1, \ldots, u_m$ ~~spans~~ is linearly independent in V.

By 2.33 of Axler, we can extend $u_1, \ldots, u_m$ to a basis $u_1, \ldots, u_m, w_1, \ldots, w_n$ of V.
Now, let $W = \text{span}(w_1, \ldots, w_n)$

To prove $V = U \oplus W$. By 1.45 of Axler, we just need to prove ~~[illegible]~~: $V = U + W$ and $U \cap W = \{0\}$
First, we will prove $V = U + W$

Suppose $v \in V$. Since $u_1, \ldots, u_m, w_1, \ldots, w_n$ spans V, there exist $a_1, \ldots, a_m, b_1, \ldots, b_m \in \mathbb{F}$ such that

$$v = \underbrace{\boxed{a_1 u_1 + \ldots + a_m u_n}}_{u} + \underbrace{\boxed{b_1 w_1 + \ldots + b_n w_n}}_{w}$$

So $v = u + w$, where $u \in U$ and $w \in W$

So $V = U + W$.

Next, we will show $U \cap W = \{0\}$.

Suppose $v \in U \cap W$. Then $u \in U$ and $u \in W$. So Then there exist $a_1, \ldots, a_m, b_1, \ldots, b_n \in \mathbb{F}$ such that

$$v = a_1 u_1 + \ldots + a_m u_m$$
$$= b_1 w_1 + \ldots + b_n w_n$$

Subtracting, we get

$$a_1 u_1 + \ldots + a_m u_m - b_1 w_1 - \ldots - b_n w_n = 0$$

Since $u_1, \ldots, u_m, w_1, \ldots, w_n$ are linearly independent, all the scalars are zero.

$$a_1 = 0, \ldots, a_m = 0, -b_1 = 0, \ldots, -b_n = 0.$$
$$(b_1 = 0, \ldots, b_n = 0)$$

~~So~~ Therefore,

$$v = a_1 u_1 + \ldots + a_m u_m$$
$$= 0u_1 + \ldots + 0u_m$$
$$= 0 \in \{0\}$$

and

$$v = b_1 w_1 + \ldots + b_n w_n$$
$$= 0w_1 + \ldots + 0w_n$$
$$= 0 \in \{0\}.$$

Therefore, $U \cap W \subset \{0\}$.

So $U \cap W = \{0\}$, as desired. $\blacksquare$

continued $\Rightarrow$

Also for example

$1, z+3, (z+3)^2, \dots, (z+3)^m$ is a basis of $P_m(\mathbb{F})$

Another ex,

$1, z-6, (z-6)^2$ is a basis of $P_2(\mathbb{F})$