7/12/2019 $\boxed{\text{2B Bases}}$

2.27 Defn

A bases of V is a list of vectors $v_1, \dots, v_m$ of V that is linearly independent and spans V

2.28 Eg

- The list $(1, 0, \cdots, 0), (0, 1, 0, \cdots, 0), \cdots, (0, \cdots, 0, 1)$ is a basis of $\mathbb{F}^n$.
  It is called the standard basis of $\mathbb{F}^n$
- $(1, 0), (0, 1)$ is a basis of $\mathbb{F}^2$
  (standard basis)
- $(1, 2), (3, 5)$ is a basis of $\mathbb{F}^2$.
- $(2, 3), (4, 6)$ is NOT a basis of $\mathbb{F}^2$
  (It is linearly dependent)
- $(1, 2, 3), (2, 1, 3)$ is NOT a basis of $\mathbb{F}^3$
  (only two vectors in $\mathbb{F}^3$, therefore does not span $\mathbb{F}^3$)
- $(1, 2), (3, 5), (4, 13)$ is NOT a basis of $\mathbb{F}^2$.
  (three vectors in $\mathbb{F}^2$, therefore one vector is linear combination of the other two)
  (linearly dependent)
- $(1, 1, 0), (0, 0, 1)$ is a basis of $\{(x, x, y) \in \mathbb{F}^3 : x, y \in \mathbb{F}\}$
  because $(x, x, y) = (x, x, 0) + (0, 0, y)$
  $= x(1, 1, 0) + y(0, 0, 1)$
- $(1, -1, 0), (1, 0, -1)$ is a basis of $\{(x, y, z) \in \mathbb{F}^3 : x + y + z = 0\}$
  ~~because $x+y+z=0$ implies $z=-x-y$. So,~~
  ~~$(x, y, z) = (x, y, -x-y)$~~
  ~~$= (x, y, -x) + (x, y, -y)$~~
  ~~$= (x, 0$~~

2.29 Criterion for Bases

A list $v_1, \cdots, v_n \in V$ is a basis of V if and only if every $v \in V$ can be written uniquely (in only one way) in the form

$$v = a_1 v_1 + \cdots + a_n v_n$$

for some $a_1, \cdots, a_n \in \mathbb{F}$.

Proof: Forward Direction

Suppose $v_1, \cdots, v_n$ is a basis of V. For all $v \in V$, there exist $a_1, \cdots, a_n$ such that

$$\star\ v = a_1 v_1 + \cdots + a_n v_n$$

We need to show that this representation is unique.

Suppose $c_1 \cdots c_n \in \mathbb{F}$ also satisfy

$$\star\ v = c_1 v_1 + \cdots + c_n v_n.$$

Subtraction $\star - \star$, we get

$$0 = (a_1 - c_1)v_1 + \cdots + (a_n - c_n)v_n.$$

Since 0 can be written in only one way, we must have:

$$a_1 - c_1 = 0, \cdots, a_n - c_n = 0.$$

Therefore,

$$a_1 = c_1, \cdots, a_n = c_n$$

and so the representation is unique.

Backward Direction

Suppose we can write every $v \in V$ uniquely in the form $v = a_1 v_1 + \cdots + a_n v_n$.

Then v is a linear combination of $v_1, \cdots, v_n$, which means $v_1, \cdots, v_n$ spans V.

Now we must prove that $v_1, \cdots, v_n$ is linearly independent. Suppose $a_1, \cdots, a_n \in \mathbb{F}$ satisfy

$$a_1 v_1 + \cdots + a_n v_n = 0$$

Since the representation of every $v \in V$ ($v = 0$, in particular) is unique, we must have
$a_1 = 0, \cdots, a_n = 0$.

Therefore, $\boxed{v_1, \cdots, v_n \text{ is linearly independent}}$

Since we proved that $v_1, \cdots, v_n$ is a linearly independent set that spans V, we conclude that $v_1, \cdots, v_n$ is a basis of V. ∎

2.31 Spanning List contains a Basis

Every spanning list is a vector space can be reduced to a basis of the vector space.

Eg $\boxed{(1,0)}, \boxed{(0,1), (2,3)}$ is a spanning list of $\mathbb{F}^2$. But we can reduce this list to $(1,0), (0,1)$.

basis of $\mathbb{F}^2$

basis of $\mathbb{F}^2$

(Proof skipped for 2.31)

2.32 Basis of finite-dimensional vector space

Every finite-dimensional vector space has a basis.

proof: According to 2.10 of Axler (Defn. of finite-dimensional vector space), there exists a spanning list of every finite-dimensional vector space.

By 2.31, the spanning list can be reduced to a basis.

2.33 Linearly Independent list Extends to a Basis

Let V be a finite-dimensional vector space, and let $u_1, \cdots, u_m$ be a linearly independent list. Then this list can be extended to a basis $w_1, \cdots, w_n$ of V

proof: Suppose $u_1, \cdots, u_m \in V$ is linearly independent and $w_1, \cdots, w_n \in V$ is a basis. extend to spanning list of V

Then the list $u_1, \cdots, u_m, w_1, \cdots, w_n$ spans V, Apply the procedure of 2.31 of Axler to remove some vectors from $u_1, \cdots, u_m, w_1, \cdots, w_n$ (if neccessary) to reduce this spanning list to a basis of V.

Eg The list (2,3,4), (2,6,8) is linearly independent in $\mathbb{F}^3$. Then following the procedure of the proof of 2.33 of Axler, we can obtain a list:

$$(2,3,4),\ (2,6,8),\ (0,1,0)$$

which is a basis of $\mathbb{F}^3$.

Exam?

2.34 Every subspace of V is part of a direct sum equal to V

Suppose V is a finite-dimensional vector space, and V is a subspace of V. Then there exists a subspace W of V such that $V = U \oplus W$

can write $v = u + w$ in <u>only one way</u>

$v \in V,\ u \in U,\ w \in W$

<u>proof</u>: Since V is finite-dimensional and U is a subspace of V, by 2.26 of Axler, U is also finite-dimensional. By 2.32 of Axler, there exist a basis $u_1, \cdots, u_m$ of U. This means in particular that $u_1, \cdots, u_m$ is linearly independent in V. By 2.33 of Axler we can extend $u_1, \cdots, u_m$ to a basis $u_1, \cdots, u_m, w_1, \cdots, w_n$ of V.

Now, let $W = \operatorname{span}(w_1, \cdots, w_n)$

To prove $V = U \oplus W$. By 1.45 of Axler, we just need to prove

$$V = U + W \text{ and } U \cap W = \{0\}.$$

- First, we prove $V = U + W$.

  Suppose $v \in V$. Since $u_1, \cdots, u_m, w_1, \cdots, w_n$ spans V, there exist $a_1, \cdots, a_m, b_1, \cdots, b_n \in \mathbb{F}$ such that

$$v = \underbrace{\boxed{a_1u_1 + \cdots + a_mu_m}}_{u} + \underbrace{\boxed{b_1w_1 + \cdots + b_nw_n.}}_{w}$$

So $v = u + w$, where $u \in U$ and $w \in W$.

proof: Next, we will show $U \cap W = \{0\}$

Suppose $v \in U \cap W$. Then $u \in U$ and $u \in W$.

So there exist $a_1, \cdots, a_m, b_1, \cdots b_n \in \mathbb{F}$ such that

$$v = a_1 u_1 + \cdots a_m u_m = b_1 w_1 + \cdots + b_n w_n$$

Subtracting, we get

$$a_1 u_1 + \cdots + a_m u_m - b_1 w_1 - \cdots - b_n w_n = 0.$$

Since $u_1, \ldots, u_m, w_1, \cdots w_n$ are linearly independent, all the scalars are zero:

$$a_1 = 0, \ldots, a_m = 0, -b_1 = 0, \ldots, -b_n = 0.$$

$$(b_1 = 0, \cdots, b_n = 0)$$

Therefore,
$$\begin{aligned} v &= a_1 v_1 + \cdots + a_m u_m \\ &= 0 u_1 + \cdots + 0 u_m \\ &= 0 \in \{0\} \end{aligned}$$

and
$$\begin{aligned} v &= b_1 w_1 + \cdots + b_n w_n \\ &= 0 w_1 + \cdots + 0 w_n \\ &= 0 \in \{0\} \end{aligned}$$

therefore, $U \cap W \subset \{0\}$

So, $U \cap W = \{0\}$, as desired.

7/3/2019 **2B Revisited**

2.28 Eg · The list $(1,-1,0), (1,0,-1)$ is a basis of

$U = \{(x,y,z) \in \mathbb{F}^3 : x+y+z=0\}$

$(x,y,z) = (x,y,-x-y)$

$= (x,0,-x) + (0,y,-y)$

$= x(1,0,-1) + (0,1,-1)$

So, $(1,0,-1), (0,1,-1)$ spans U.

Suppose $a, b \in \mathbb{F}$ satisfy

$a(1,0,-1) + b(0,1,-1) = (0,0,0)$

$(a,0,-a) + (0,b,-b) = (0,0,0)$

$(a,b,-a-b) = (0,0,0)$

Equate coordinates

$a=0$

$b=0$

$-a-b=0$

Therefore, the list $(1,0,-1), (0,1,-1)$ is linearly independent.

So, $(1,0,-1), (0,1,-1)$ is a basis of U.

The list $1, z, \cdots, z^m$ is a basis of $P_m(\mathbb{F})$.

Every polynomial $p \in P_m(\mathbb{F})$ is written

$p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_m z^m$ for all $\boxed{z \in \mathbb{F}}$,

for all $a_0, a_1, a_2, \cdots, a_m \in \mathbb{F}$. defn 2.12

Note that $p(z)$ is a linear combination of

$1, z, z^2, \cdots z^m$.

Eg Also for example

$1, z+3, (z+3)^2, \cdots, (z+3)^m$.

is a basis of $P_m(\mathbb{F})$

Eg Another example

$1, z-6, (z-6)^2$

is a basis of $P_2(\mathbb{F})$